THE SIXTH REGULAR MEETING OF THE SOUTHWESTERN SECTION.

THE sixth regular meeting of the Southwestern Section of the Society was held at the University of Kansas, Lawrence, Kansas, on Saturday, November 30, 1912. About thirty persons were present including the following nineteen members of the Society:

Professor C. H. Ashton, Dr. Elizabeth R. Bennett, Professor W. C. Brenke, Professor E. W. Davis, Dr. Otto Dunkel, Professor E. P. R. Duval, Professor R. R. Fleet, Professor W. H. Garrett, Professor W. A. Harshbarger, Professor E. R. Hedrick, Dr. S. Lefschetz, Miss Hazel H. MacGregor, Professor U. G. Mitchell, Dr. Mary W. Newson, Professor S. W. Reaves, Professor W. H. Roever, Professor G. H. Scott, Professor J. N. Van der Vries, Professor Marion B. White.

The morning session was held at 10 A. M. and the afternoon session at 2:30 P. M. Professor Van der Vries presided. It was decided to hold the next meeting of the section at Columbia, Missouri, on November 29, 1913, and the following programme committee was elected: Professors Hedrick (chairman), Brenke, and Kellogg (secretary).

The members present were the guests at a smoker and reception given on the evening before the meeting at the Alpha Tau Omega chapter house.

The following papers were presented at this meeting:

(1) Professor W. H. ROEVER: "The design and theory of a mechanism for illustrating certain systems of lines of force and stream lines."

(2) Professor S. W. REAVES: "On the projective differential geometry of plane anharmonic curves."

(3) Dr. E. R. BENNETT: "Transitive groups of degree 107."

(4) Dr. E. R. BENNETT: "The order of the product of two substitutions."

(5) Professor ARNOLD EMCH: "On closed continuous curves."

(6) Professor R. D. CARMICHAEL: "On the theory of relativity: mass, energy, gravitation; philosophical aspects."

(7) Professor FLORIAN CAJORI: "Historical note on the graphic representation of imaginaries before the time of Wessel."

(8) Dr. S. LEFSCHETZ: "Geometry on ruled surfaces."
(9) Dr. E. L. DODD: "The error-risk of certain functions of the measurements."
(10) Dr. T. H. GRONWALL: "On the Riemann zeta function (first paper)."
(11) Dr. T. H. GRONWALL: "On the Riemann zeta function (second paper)."
(12) Dr. T. H. GRONWALL: "On Dirichlet's series corresponding to complex characters."
(13) Dr. T. H. GRONWALL: "On Picard's theorem."
(14) Professor E. R. HEDRICK: "Direct definition of the logarithmic derivative."
(15) Professor LOUIS INGOLD: "Note on systems of integral equations."

In the absence of the authors, the papers of Professor Emch, Professor Carmichael, Dr. Dodd, and Dr. Gronwall were read by title, the paper of Professor Cajori was presented by Professor Mitchell, and the paper of Professor Ingold by Professor Hedrick. Abstracts of the papers follow below.

1. At the Chicago meeting (April 5–6, 1912) Professor Roever exhibited a mechanism for illustrating certain systems of lines of force. This consisted essentially of two wheels with radial spokes which could be made to rotate in nearly coincident planes and thus render visible the loci of the intersections of the spokes (BULLETIN, volume 18, page 435). Professor Roever now shows that not only for radial spokes but also for spokes in the form of logarithmic spirals, the mechanism illustrates lines of force. By having several pairs of wheels with different systems of logarithmic spirals (in particular, of radial straight lines) it is possible, in virtue of the device for obtaining different angular velocity ratios, to obtain a great number of different systems of curves, each of which may be regarded as lines of force or stream lines in five or six different branches of mathematical physics. It is possible, by making time exposures, to obtain well-defined photographs of these systems of curves. This possibility adds to the usefulness of the mechanism. The mechanism and photographs of some of the systems of curves were exhibited.

2. In this paper Professor Reaves applies the theory of plane curves as developed by Halphen and Wilczynski to a

study of certain projective differential properties of the anharmonic curves

$$y = x^{\lambda}, \quad y = e^{x}, \quad \text{and} \quad r = e^{m\theta}.$$

For each of these curves are found the osculating conic at an arbitrary point and the locus of its center; the Halphen point and its locus; the point in which the tangent meets the straight line on which lie the three points of inflection of the eight-pointic nodal cubic of the point of contact, and the locus of this point. For each curve these loci are found to be curves of the same type and having the same invariant triangle as the curve from which they are derived. The osculating conic of the logarithmic spiral is studied in some detail and a construction is given for its center and axes.

3. Transitive groups of degree $p = 2q + 1$, p and q being prime numbers, have been considered especially by Mathieu, de Séguier, and Miller and all groups of this type, $p < 107$, have been determined. Professor Miller's method for the determination of these groups is applied by Miss Bennett to the determination of the transitive groups of degree 107. It is shown that only the six well-known transitive groups of this degree exist.

4. Miss Bennett determined the order of the product of two circular substitutions having common elements when the arrangement of the common elements satisfied certain conditions.

5. In a paper soon to be published, Professor Emch shows that in every closed continuous curve at least one square may be inscribed. The object of the present communication is the solution of the inverse problem: to find the parametric equations of all continuous curves through the vertices of any square in the plane.

Designating by t a parameter, the equations of such a curve may always be written in the form

$$x = p + \frac{a}{2}\sqrt{2}\cos\left(\frac{2\pi t}{\omega} + \theta\right) + \prod_{k=0}^{3}\sin\left\{\frac{2\pi t}{\omega} - (2k+1)\frac{\omega}{8}\right\} F(t), \tag{1}$$

$$(1)\qquad y = q + \frac{a}{2}\sqrt{2}\sin\left(\frac{2\pi t}{\omega} + \theta\right) + \prod_{k=0}^{3}\sin\left\{\frac{2\pi t}{\omega} - (2k+1)\frac{\omega}{8}\right\}G(t),$$

where $F(t)$ and $G(t)$ are single valued continuous functions of t. If these are periodic (with the same period ω), the curve represented by (1) is a closed curve. Every closed continuous curve through the vertices of a square may therefore be represented in the form (1).

If it is possible to show that the parametric equations of any closed curve

$$(2)\qquad x = \phi(t'), \quad y = \psi(t'),$$

where $\phi(t')$ and $\psi(t')$ are any continuous coperiodic functions of t', by proper transformation of t' into t, can be reduced to (1), then every closed curve admits of at least one inscribed square.

6. Professor Carmichael's paper on relativity falls into two parts. The first part deals with the theory of mass, energy and gravitation. The characteristic conclusions of the theory of relativity are derived in a simple manner. These are then applied to the theory of the Kaufmann and Bucherer experiments concerning the ratio of electric charge to the mass of a moving charged body. There is then a discussion of the bearing of these experiments on the general question of the experimental proof of the theory of relativity. The second part of the paper contains a discussion of some philosophical questions connected with that part of the theory of relativity which is developed in this paper and in an earlier one by Professor Carmichael (see *Physical Review*, September, 1912, pages 153–176).

7. Professor Cajori refers to a letter of Wallis to Collins May 6, 1673, which gives Wallis's earliest attempts at visualization of imaginaries. There is ground for suspecting that the Wessel-Argand diagram was known earlier to Euler, Walmesley, and others, and that Gauss received suggestions from Euler. The most important contribution before Wessel is the diagram of W. J. G. Karsten, 1768, which exhibits the infinitely many logarithms of real and complex numbers.

8. In this paper Dr. Lefschetz gives two proofs of a formula given by W. E. Story in 1883, applying to scrolls of any order μ, for the number of intersections of curves of order a, b meeting the generators in α and β points. One of the proofs is based on a previous work of Severi, and the other, of an elementary character, on the theory of elimination.

9. Dr. Dodd uses the conception of the error-risk* to draw comparisons between the arithmetic mean and certain functions of the measurements,—the error-risk being a species of probable value. The following theorems are closely related to the theorems presented in a paper at the recent summer meeting.

The Gaussian Law [G. L.] is postulated in this form: viz., that the probability that the error x of the measurement m will lie in the interval (x', x'') is given by

$$\frac{h}{\sqrt{\pi}}\int_{x'}^{x''} e^{-h^2x^2}dx,$$

where $x' < x''$.

The "true value" is designated by a, and the arithmetic mean of n measurements by M_n, or simply M.

Theorem. Under the G. L., the probable value of the square of the error of M is greater than that of BM, if B is any constant such that

$$1 - \frac{2}{2nh^2a^2+1} < B < 1. \tag{1}$$

Usually ha is very large. But, if not, then from the standpoint of the "mean error-risk," it is *useless to make any measurements* at all—with the purpose of taking the arithmetic mean—*unless* a sufficient number, n, are made so that

$$2nh^2a^2 \geqq 1. \tag{2}$$

For, if (2) is not satisfied, the risk in accepting zero as the value of the unknown is less than that in accepting M.

A somewhat similar theorem is valid, involving the absolute value of the error of M, and another involving any even power of the error of M.

Now let

$$M_n' = + \sqrt{\frac{m_1^2 + m_2^2 + \cdots + m_n^2}{n}},$$

*"Das Fehlerrisiko," Czuber, Wahrscheinlichkeitsrechnung, I, p. 266.

in particular,

$$M_2' = +\tfrac{1}{2}\sqrt{\frac{m_1^2 + m_2^2}{2}}.$$

Theorem. Under the G. L., the probable value of the square of the error of M_2' is less than that of M_2, if $a > 0$, and h is sufficiently great.

Theorem. Under the G. L., the probable value of the square of the error of M_n' is greater than that of M_n if $n > 3$, and h is sufficiently great.

By a weighted mean W is meant $p_1m_1+p_2m_2+\cdots+p_nm_n$, where the p's are constants whose sum is unity; and by an absolute error an error taken positively.

Theorem. If the measurements are subject to the G. L. with measures of precision $h_1, h_2, \cdots, h_n$, respectively, then the probable value of any given positive power of the absolute error of W is least when $p_i = h_i^2/\Sigma h^2$. In particular, if the h's are all equal, this W with the least error-risk is M.

10. Let $s = \sigma + ti$ be a complex variable; for $\sigma > 1$, the Riemann zeta function is defined by the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

which ceases to converge for $\sigma = 1$. On this line, and for $|t|$ sufficiently large, the following upper and lower limitations for the zeta function were found by Mellin and Landau:

$$|\zeta(1 + ti)| < \log|t| + c_1,$$

$$\frac{1}{|\zeta(1 + ti)|} < c_2 \log|t| \cdot \log\log|t|,$$

where c_1 and c_2 are positive constants.

In the present paper, Dr. Gronwall proves, by relatively elementary means, the closer limitations

$$|\zeta(1 + ti)| < \tfrac{3}{4}\log|t| + c_1,$$

$$\frac{1}{|\zeta(1 + ti)|} < c_2 \log|t| \cdot (\log\log|t|)^{\frac{3}{4}}.$$

11. In continuation of the investigations of the preceding paper, and using some of the less elementary properties of the zeros of the zeta function, Dr. Gronwall finds that

$$\frac{1}{|\zeta(1+ti)|} < c_2 \log|t|,$$
$$\left|\frac{\zeta'(1+ti)}{\zeta(1+ti)}\right| < c_3 \log|t|.$$

12. In showing that any arithmetical progression $kx+l$, where k and l are relative primes, contains an infinity of prime numbers, Dirichlet introduced the series

$$L_\kappa(s) = \sum_{n=1}^{\infty} \frac{\chi_\kappa(n)}{n^s},$$

where $\chi_\kappa(n)$ is a certain kth root of unity called the character of n. The main point in Dirichlet's investigation consisted in showing that $L_\kappa(1) \neq 0$. For any complex character, Landau has recently obtained the limitation

$$\frac{1}{|L_\kappa(1)|} < c \cdot (\log k)^5,$$

where c is independent of k and l.

In the present paper, Dr. Gronwall derives the closer limitation

$$\frac{1}{|L_\kappa(1)|} < c \cdot \log k \cdot (\log\log k)^2.$$

13. According to Landau's extension of Picard's theorem, there corresponds to any given power series

$$a_0 + a_1 x + a_2 x^2 + \cdots$$

a minimum radius $\varphi(a_0, a_1)$, depending on the first two coefficients only, and such that inside or on the circle $|x| = \varphi(a_0, a_1)$ the series either assumes one of the values 0 and 1, or has a singular point. The exact expression for this radius was obtained by Carathéodory in terms of the modular function, while the ordinary elementary methods for proving the theorem in question only give an upper limit for the radius of a much higher order of magnitude.

In the present paper, Dr. Gronwall obtains, by quite elementary considerations, upper and lower limits for this radius, each of which is of the same order of magnitude as the exact expression.

14. In this paper Professor Hedrick suggests several possible direct definitions of the logarithmic derivative, and discusses the effect of each of them upon certain fundamental theorems and upon the existence of the concept at points where the given function vanishes.

15. In this note Professor Ingold gives the general solution for φ of the system of equations

$$\int_a^\beta a_i(x)\varphi(x)dx = b_i$$

both for the case $i=1, 2, \cdots, n$ and the case $i=1, 2, \cdots, \infty$. The result is there used to obtain a solution of the equation

$$f(s) = \int_a^\beta K(s, t)\varphi(t)dt$$

when the kernel can be expressed as a uniformly convergent series

$$K(s, t) = \Sigma a_i(s)u_i(t)$$

and f may be written as a convergent series

$$f(s) = \Sigma c_i a_i(s).$$

Neither of the systems $a_i(s)$, $u_i(s)$ is assumed to be orthogonal, or to be closed; see also Picard, *Rendiconti del Circolo Matematico*, volume 29.

JOHN N. VAN DER VRIES,
Secretary (pro tempore) of the Section.

SOME SPECIAL BOUNDARY PROBLEMS IN THE THEORY OF HARMONIC FUNCTIONS.

BY DR. T. H. GRONWALL.

(Read before the American Mathematical Society, September 11, 1912.)

1. In his paper on "Le problème de Dirichlet dans une aire annulaire," *Rendiconti del Circolo Matematico di Palermo*, volume 33 (1912), pages 134–174, H. Villat has shown that when $u(\rho, \theta)$ is harmonic, uniform, and regular in the circular ring $R > \rho > r$, and subject to the boundary conditions

$$u(R, \theta) = \Phi(\theta), \qquad u(r, \theta) = \Psi(\theta), \tag{1}$$

$\Phi(\theta)$ and $\Psi(\theta)$ being integrable for $0 \leqq \theta \leqq 2\pi$ and satisfying the condition

$$\int_0^{2\pi} \Phi(\theta)d\theta = \int_0^{2\pi} \Psi(\theta)d\theta \tag{2}$$

necessary for the uniformity of $u(\rho, \theta)$, then this function is given by

$$\begin{aligned} u(\rho,\theta)+iv(\rho,\theta) = {} & \frac{i\omega}{\pi^2}\int_0^{2\pi} \Phi(\alpha)\frac{\sigma'}{\sigma}\left(\frac{\omega}{i\pi}\log\frac{\rho}{R}+\frac{\omega}{\pi}(\theta-\alpha)\right)d\alpha \\ & -\frac{i\omega}{\pi^2}\int_0^{2\pi} \Psi(\alpha)\frac{\sigma_3'}{\sigma_3}\left(\frac{\omega}{i\pi}\log\frac{\rho}{R}+\frac{\omega}{\pi}(\theta-\alpha)\right)d\alpha, \end{aligned} \tag{3}$$

where σ and σ_3 are the Weierstrass elliptic sigma functions, the periods ω and ω' satisfying the conditions

$$\omega \text{ and } \frac{\omega'}{i} \text{ real}, \quad \frac{\omega'}{i\omega} = \frac{1}{\pi}\log\frac{R}{r}.$$

Villat's proof consists in a somewhat lengthy discussion of the integrals in (3), similar to the ordinary treatment of Poisson's integral.

Fejér* has solved Dirichlet's problem for a circle without the use of Poisson's integral. In the present note, I propose

* L. Fejér, "Untersuchungen über trigonometrische Reihen," *Math. Annalen*, vol. 58 (1904), pp. 51–69.

to extend Fejér's process to obtain Villat's results (and the corresponding ones in three-dimensional space) in a considerably shorter way than by the method referred to.

2. We first notice that, $\Phi(\theta)$ and $\Psi(\theta)$ being integrable, their Fourier coefficients

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \Phi(\alpha)\cos n\alpha\, d\alpha, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} \Phi(\alpha)\sin n\alpha\, d\alpha,$$
$$a_n' = \frac{1}{\pi}\int_0^{2\pi} \Psi(\alpha)\cos n\alpha\, d\alpha, \quad b_n' = \frac{1}{\pi}\int_0^{2\pi} \Psi(\alpha)\sin n\alpha\, d\alpha \tag{4}$$

exist, which fact we denote by

$$\Phi(\theta) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos n\theta + b_n\sin n\theta),$$
$$\Psi(\theta) \sim \frac{a_0'}{2} + \sum_{n=1}^{\infty}(a_n'\cos n\theta + b_n'\sin n\theta), \tag{5}$$

the equivalence sign $\sim$ implying no statement whatever in regard to the convergence of the right-hand members. We now assume a development

$$u(\rho,\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty}(A_n\cos n\theta + B_n\sin n\theta)\rho^n + \sum_{n=1}^{\infty}(C_n\cos n\theta + D_n\sin n\theta)\rho^{-n}, \tag{6}$$

make $\rho = R$ and $\rho = r$, and identify the resulting Fourier series, term by term, with the Fourier series (5) for $\Phi(\theta)$ and $\Psi(\theta)$ respectively, which gives the equations

$$A_0 = a_0 = a_0',$$
$$A_nR^n + C_nR^{-n} = a_n, \quad B_nR^n + D_nR^{-n} = b_n,$$
$$A_nr^n + C_nr^{-n} = a_n', \quad B_nr^n + D_nr^{-n} = b_n' \tag{7}$$
$$(n = 1, 2, 3, \cdots).$$

The two equations for A_0 are consistent by (2), and from the others we obtain, making $r = qR$, so that $q < 1$,

$$A_n = \frac{R^{-n}}{q^{-n} - q^n}(a_nq^{-n} - a_n'), \quad B_n = \frac{R^{-n}}{q^{-n} - q^n}(b_nq^{-n} - b_n'),$$
$$C_n = \frac{R^n}{q^{-n} - q^n}(a_n' - a_nq^n), \quad D_n = \frac{R^n}{q^{-n} - q^n}(b_n' - b_nq^n). \tag{8}$$

To investigate the convergence of (6), we note that, when n is large enough to make $q^{2n} < \frac{1}{2}$,

$$
\begin{aligned}
|A_n| &= R^{-n}\left|\frac{a_n - a_n'q^n}{1-q^{2n}}\right| < 2R^{-n}(|a_n| + |a_n'|),\\
|B_n| &= R^{-n}\left|\frac{b_n - b_n'q^n}{1-q^{2n}}\right| < 2R^{-n}(|b_n| + |b_n'|),\\
|C_n| &= R^nq^n\left|\frac{a_n' - a_nq^n}{1-q^{2n}}\right| < 2r^n(|a_n| + |a_n'|),\\
|D_n| &= R^nq^n\left|\frac{b_n' - b_nq^n}{1-q^{2n}}\right| < 2r^n(|b_n| + |b_n'|).
\end{aligned}
\tag{9}
$$

Now, $\Phi(\theta)$ and $\Psi(\theta)$ being integrable, $\frac{1}{n}a_n, \frac{1}{n}b_n, \frac{1}{n}a_n', \frac{1}{n}b_n'$ are bounded for all values of n,* and therefore it follows from (9) that both series in (6), together with their partial derivatives term by term to any order, are uniformly convergent, the first for $\rho \leqq R - \epsilon$, $0 \leqq \theta \leqq 2\pi$, and the second for $\rho \geqq r + \epsilon$, $0 \leqq \theta \leqq 2\pi$, where ϵ is positive and as small as we like.

Hence the expression (6) is harmonic, uniform, and regular for $R > \rho > r$. To show that the boundary conditions (1) are satisfied, we shall use the following lemma:†

Let

$$F(x, \theta) = \sum_{n=0}^{\infty} u_n(\theta)x^n, \tag{10}$$

and suppose that the (eventually divergent) series

$$\sum_{n=0}^{\infty} u_n(\theta)$$

is summable by Cesàro's mean values of the first order, with the sum $s(\theta)$ (that is, the limit

$$\lim_{n=\infty} \frac{1}{n+1}\sum_{\mu=0}^{n}\sum_{\nu=0}^{\mu} u_\nu(\theta) = s(\theta) \tag{11}$$

* Fejér, l. c.

† Fejér, l. c., § 2, specializing his general theorem by making $t = \log(1/x)$, $\varphi(t) = e^{-t}$. Compare also G. H. Hardy, "Some theorems concerning infinite series," *Math. Annalen*, vol. 64 (1907), pp. 77–94.

does exist); then (10) is convergent for $0 \leqq x < 1$ and

$$\lim_{h=+0} F(1-h,\ \theta) = s(\theta), \tag{12}$$

and on any range for θ where (11) holds uniformly the same is true of (12).

Now the Fourier series (5) for $\Phi(\theta)$ is uniformly summable of the first order with the sum $\Phi(\theta)$ on any range where $\Phi(\theta)$ is continuous.* Furthermore, the series

$$\Phi_1(\theta) = \sum_{n=1}^{\infty} (C_n \cos n\theta + D_n \sin n\theta) R^{-n},$$

being uniformly convergent, is obviously uniformly summable of the first order with the sum $\Phi_1(\theta)$; hence, by (7), the series

$$\frac{A_0}{2} + \sum_{n=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta) R^n$$

is uniformly summable of the first order with the sum $\Phi(\theta) - \Phi_1(\theta)$. Making $x = \rho/R$ and applying the lemma, it is seen that

$$\lim_{h=+0} u(R(1-h),\ \theta) = \Phi(\theta) - \Phi_1(\theta) + \Phi_1(\theta) = \Phi(\theta) \tag{13}$$

uniformly on any range where $\Phi(\theta)$ is continuous. Making $x = r/\rho$ and reasoning in the same way on $\Psi(\theta)$, we obtain

$$\lim_{h=+0} u\left(\frac{r}{1-h},\ \theta\right) = \Psi(\theta) \tag{14}$$

uniformly on any range where $\Psi(\theta)$ is continuous, so that our expression (6) satisfies all the conditions of the problem.

Introducing the values (4) into (8) and using well known formulæ in elliptic functions, we may easily reduce (6) to the form (3).

3. Now suppose that $\Phi(\theta)$ is continuous for $\theta = \theta_0$, and let the point $\rho,\ \theta$ approach $R,\ \theta_0$ along any continuous path $\rho = \rho(\theta)$ instead of the radius. For any positive ϵ we may find a δ such that, $\Phi(\theta)$ being continuous for $\theta = \theta_0$,

$$|\Phi(\theta) - \Phi(\theta_0)| < \tfrac{1}{2}\epsilon \text{ for } \theta_0 - \delta < \theta < \theta_0 + \delta.$$

* Fejér, l. c., § 1.

By (13), there exists an η independent of θ such that

$$| u(R(1-h), \theta) - \Phi(\theta) | < \tfrac{1}{2}\epsilon$$
$$\text{for } 0 < h \leqq \eta \text{ and } \theta_0 - \delta < \theta < \theta_0 + \delta,$$

and according to the definition of $\rho(\theta)$ there exists a $\delta' < \delta$ such that

$$0 < R - \rho(\theta) < R\eta \text{ for } \theta_0 - \delta' < \theta < \theta_0 + \delta'.$$

The combination of these inequalities gives

$$| u(\rho(\theta), \theta) - \Phi(\theta_0) | < \epsilon \text{ for } \theta_0 - \delta' < \theta < \theta_0 + \delta',$$

so that, for any θ_0 where $\Phi(\theta)$ is continuous, $u(\rho, \theta)$ tends towards $\Phi(\theta_0)$ when ρ, θ approaches R, θ_0 by any continuous path entirely inside the circle of radius R.

The case of a point of discontinuity θ_0, such that $\Phi(\theta_0 + 0)$ and $\Phi(\theta_0 - 0)$ both exist, may be reduced to the continuity case, according to H. A. Schwarz, by replacing $u(\rho, \theta)$ by

$$u(\rho, \theta) - \frac{\Phi(\theta_0 + 0) - \Phi(\theta_0 - 0)}{\pi} \operatorname{arctg} \frac{\rho \sin \theta - R \sin \theta_0}{\rho \cos \theta - R \cos \theta_0},$$

and from the preceding results, all of Villat's theorems are easily deduced.

4. Passing to three-dimensional space referred to polar coordinates, consider the harmonic function $u(\rho, \theta, \varphi)$, uniform and regular for $r < \rho < R$, and with the boundary conditions

$$u(R, \theta, \varphi) = \Phi(\theta, \varphi), \quad u(r, \theta, \varphi) = \Psi(\theta, \varphi),$$

where Φ and Ψ are integrable on the unit sphere.

We then have the formal developments in spherical harmonics

$$(15) \quad \Phi(\theta, \varphi) \backsim \sum_{n=0}^{\infty} Y_n'(\theta, \varphi), \quad \Psi(\theta, \varphi) \backsim \sum_{n=0}^{\infty} Y_n''(\theta, \varphi),$$

and assuming a development

$$(16) \quad u(\rho, \theta, \varphi) = Y_0 + \sum_{n=1}^{\infty} Y_n(\theta, \varphi)\rho^n + \sum_{n=1}^{\infty} \overline{Y}(\theta, \varphi)\rho^{-n},$$

we make $\rho = R$ and $\rho = r$ and identify (16) term by term

with the developments of Φ and Ψ respectively, obtaining

$$Y_0 = Y_0' = Y_0' \quad \text{(condition of uniformity)},$$
$$R^n Y_n + R^{-n}\bar{Y}_n = Y_n',$$
$$r^n Y_n + r^{-n}\bar{Y}_n = Y_n'' \qquad (n = 1, 2, 3, \cdots),$$

whence

$$(17)\qquad \begin{aligned} Y_n &= \frac{R^{-n}}{q^{-n} - q^n}(q^{-n}Y_n' - Y_n''), \\ \bar{Y}_n &= \frac{q^n R^n}{q^{-n} - q^n}(q^{-n}Y_n'' - Y_n'), \end{aligned} \qquad (n = 1, 2, 3, \cdots).$$

By expressing Y_n' and Y_n'' by finite trigonometric sums in the usual way and slightly modifying Fejér's argument for showing the boundedness of $a_n/n, \cdots, b_n'/n$ in (8), it is readily seen that Y_n'/n^4 and Y_n''/n^4 are bounded, and the convergence of (16) and its partial derivatives term by term is then established exactly as that of (6). The expression (16) is therefore harmonic, uniform, and regular for $r < \rho < R$; to prove that the boundary conditions are satisfied, we may proceed in the same way as before, only substituting Cesàro's means of the second order for those of the first order previously used.*

Introducing the well-known integral expressions for Y_n' and Y_n'', we may write (16) in the form

$$(18)\qquad \begin{aligned} u(\rho, \theta, \varphi) = \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} \Phi(\alpha, \beta)\left\{1 + \sum_{n=1}^{\infty}\frac{2n+1}{q^{-n}-q^n}\left[\left(\frac{\rho}{qR}\right)^n\right.\right. \\ \left.\left. - \left(\frac{\rho}{qR}\right)^{-n}\right]P_n(\cos\gamma)\right\}\sin\alpha\, d\alpha d\beta \\ - \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}\Psi(\alpha, \beta)\sum_{n=1}^{\infty}\frac{2n+1}{q^{-n}-q^n}\left[\left(\frac{\rho}{R}\right)^n\right. \\ \left. - \left(\frac{\rho}{R}\right)^{-n}\right]P_n(\cos\gamma)\sin\alpha\, d\alpha d\beta, \end{aligned}$$

* In the general case of summability of $\sum_{n=0}^{\infty} u_n(\theta)$ by Cesàro's means of the rth order, the lemma was proved by Hardy, l. c. The summability of the second order of the development of a function $f(\theta, \varphi)$ in spherical harmonics was proved by Fejér, "Ueber die Laplacesche Reihe," *Math. Annalen*, vol. 67 (1909), pp. 76–109, when $f(\theta, \varphi)$ is *absolutely* integrable, and by the present writer (in a paper with the same title which will appear presently in the *Math. Annalen*), when $(f(\theta, \varphi)$ is any integrable fnnction. In the same paper, it is shown that for an absolutely integrable function, already the means of the first order are summable.

γ being the angle between the directions θ, φ and α, β. By using the Mehler formulæ for Legendre's polynomials P_n, (18) may be transformed so as to contain elliptic sigma functions under a triple integral sign, giving a formula somewhat similar to (3).

CHICAGO, ILL.,
November 9, 1912.

NOTE ON FERMAT'S LAST THEOREM.

BY PROFESSOR R. D. CARMICHAEL.

(Read before the American Mathematical Society, December 31, 1912.)

THE object of this note is to prove the following

THEOREM. *If p is an odd prime and the equation*

$$x^p + y^p + z^p = 0 \tag{1}$$

has a solution in integers x, y, z each of which is prime to p, then there exists a positive integer s, less than $\frac{1}{2}(p-1)$, such that

$$(s+1)^{p^2} \equiv s^{p^2} + 1 \bmod p^3.$$

The proof is elementary. If there exists a set of integers x, y, z satisfying (1), there exists such a set having the further property that they are prime each to each. Consequently, for the purpose of argument we may assume that x, y, z have this property.

Then from elementary considerations it is known* that integers α, β, γ exist such that

$$x + y = \gamma^p, \quad y + z = \alpha^p, \quad z + x = \beta^p.$$

Therefore

$$(2) \quad (x+y)^{p-1} \equiv 1, \quad (y+z)^{p-1} \equiv 1, \quad (z+x)^{p-1} \equiv 1 \bmod p^2,$$

since $a^{p(p-1)} \equiv 1 \bmod p^2$ when a is prime to p.

From (1) it follows that

$$x + y + z \equiv 0 \bmod p,$$

* See, for instance, Bachmann's Niedere Zahlentheorie, Zweiter Teil, p. 467.

since $x^p \equiv x$, $y^p \equiv y$, $z^p \equiv z \bmod p$ by Fermat's theorem. Writing $x + y = Ap - z$, we have readily $(x + y)^p \equiv -z^p \bmod p^2$. Replacing $-z^p$ by its value $x^p + y^p$ and writing the resulting congruence and two similar ones, we have

$$(3) \qquad \begin{gathered}(x + y)^p \equiv x^p + y^p, \quad (y + z)^p \equiv y^p + z^p, \\ (z + x)^p \equiv z^p + x^p \bmod p^2.\end{gathered}$$

From (2) and (3) we see that

$$x^p + y^p \equiv x + y, \quad y^p + z^p \equiv y + z, \quad z^p + x^p \equiv z + x \bmod p^2.$$

Adding these congruences and making use of equation (1), we have

$$x + y + z \equiv 0 \bmod p^2.$$

Then we may write $x + y = Bp^2 - z$, whence $(x + y)^p \equiv -z^p \bmod p^3$. Hence, since $-z^p = x^p + y^p$, we have

$$\begin{gathered}(x + y)^p \equiv x^p + y^p, \quad (y + z)^p \equiv y^p + z^p, \\ (z + x)^p \equiv z^p + x^p \bmod p^3.\end{gathered}$$

Adding these three congruences and employing (1), we have

$$(4) \qquad (x + y)^p + (y + z)^p + (z + x)^p \equiv 0 \bmod p^3.$$

Now from (2) we have $(x + y)^{p-1} = 1 + cp^2$, whence it follows that

$$(x + y)^{p(p-1)} \equiv 1 \bmod p^3; \text{ or } (x + y)^{p^2} \equiv (x + y)^p \bmod p^3,$$

with similar congruences for $y + z$ and $z + x$. From these congruences and (4) we have the following relation:

$$(5) \qquad (x + y)^{p^2} + (y + z)^{p^2} + (z + x)^{p^2} \equiv 0 \bmod p^3.$$

But $x + y = Ap - z$, and therefore $(x + y)^{p^2} \equiv -z^{p^2} \bmod p^3$, with similar congruences for $y + z$ and $z + x$. Substituting in (5), we have

$$(6) \qquad x^{p^2} + y^{p^2} + z^{p^2} \equiv 0 \bmod p^3.$$

Now let σ be the positive integer less than p such that

$$y \equiv \sigma x \bmod p.$$

Then since $x+y+z\equiv 0 \bmod p$ we have $x+\sigma x+z\equiv 0 \bmod p$ or $z\equiv -(\sigma+1)x \bmod p$. It is then obvious that $y^{p^2}\equiv(\sigma x)^{p^2} \bmod p^3$ and $z^{p^2}\equiv\{-(\sigma+1)x\}^{p^2} \bmod p^3$. Substituting in (6) and dividing the resulting congruence by x^{p^2}, we have

$$(\sigma+1)^{p^2}\equiv\sigma^{p^2}+1 \bmod p^3. \tag{7}$$

If $\sigma<\frac{1}{2}(p-1)$, it may be taken for the s of the theorem, and our demonstration is then complete. If $\sigma>\frac{1}{2}(p-1)$, write

$$s=p-\sigma-1,$$

whence $s<\frac{1}{2}(p-1)$. From (7) we have

$$(\sigma+1-p)^{p^2}\equiv(\sigma-p)^{p^2}+1 \bmod p^3;$$

or

$$(p-\sigma)^{p^2}\equiv(p-\sigma-1)^{p^2}+1 \bmod p^3,$$

whence

$$(s+1)^{p^2}\equiv s^{p^2}+1 \bmod p^3.$$

If $\sigma=\frac{1}{2}(p-1)$ we have from (7), on multiplying by 2^{p^2},

$$(p+1)^{p^2}\equiv(p-1)^{p^2}+2^{p^2} \bmod p^3;$$

or

$$1\equiv -1+2^{p^2} \bmod p^3,$$

whence

$$2^{p^2}\equiv 2 \bmod p^3,$$

so that for the s of the theorem we may in this case take $s=1$. This completes the demonstration of the theorem.

COROLLARY. *If any two of the numbers x, y, z are congruent modulo p, then*

$$2^{p^2-1}\equiv 1 \bmod p^3.$$

For, if x and y are congruent, σ is equal to unity and the corollary follows at once from (7). A similar proof may of course be made when y and z or z and x are congruent.

From the way in which the theorem was proved it is clear that in general there are several values of σ satisfying congruence (7). Thus if $\sigma\tau\equiv 1 \bmod p$, so that $x\equiv\tau y \bmod p$,

it is obvious that we have also

$$(\tau + 1)^{p^2} \equiv \tau^{p^2} + 1 \mod p^3.$$

But this congruence is implied by (7) alone, as one may readily verify by multiplying (7) by τ^{p^2}. Other cases may be dealt with similarly.

INDIANA UNIVERSITY,
November, 1912.

INTEGRAL EQUATIONS.

Introduction à la Théorie des Equations intégrales. By T. LALESCO. Paris, A. Hermann et Fils, 1912. 152 pp.

L'Equation de Fredholm et ses Applications à la Physique mathématique. By H. B. HEYWOOD and M. FRECHET. Paris, A. Hermann et Fils, 1912. 165 pp.

THE theory of integral equations has been developed since the publication of Volterra's first paper in 1896, and most of the work has been done since Fredholm's fundamental memoir appeared in 1900. Yet, in this comparatively short time, the number of printed papers dealing with the subject has become so great that one approaching the subject for the first time is embarrassed by the wealth of material at his command. The two books mentioned above have been written for the beginner in the study of this interesting and useful branch of analysis. The authors have given a clear and concise exposition of the fundamental principles and of the most important results obtained up to the present time. While admitting freely that there is much yet to be done both on the theoretical side and the side of applications to mathematical physics and mechanics, there can be no doubt that the fundamental portions have already reached a form that will remain classic, and that it is now desirable to have them in book form for the convenience of the mathematical public. These two small volumes will be found very useful to the reader who wishes merely an acquaintance with the first principles of the subject, as well as to the reader who expects to attain a wider knowledge by studying the journal articles. While there is necessarily some repetition the two books may well be used together. The first one is devoted to the theory of integral equations and

makes no mention of applications, while the second is concerned chiefly with the application of Fredholm's equation to the problems of mathematical physics.

The treatise of Lalesco is divided into three parts entitled: I. The Equation of Volterra, II. The Equation of Fredholm, and III. Singular Equations. Beginning in part I with the equation of Volterra of the first kind

$$(1) \qquad \int_0^x N(x, s)\varphi(s)ds = F(x) \qquad (0 \leqq x \leqq a),$$

where $F(x)$ and the kernel $N(x, s)$ are known functions while φ is to be determined, the author first shows the analogy of the theory of linear integral equations to that of a system of linear algebraic equations. Throughout this part of the work it is assumed that we have to deal only with finite and integrable functions. Differentiating (1) with respect to x and assuming that $N(x, x)$ does not vanish in the interval $(0a)$ leads to an equation of the second kind, that is, of the form

$$(2) \qquad \varphi(x) + \int_0^x N(x, s)\varphi(s)ds = F(x).$$

To solve this equation by successive approximations a parameter λ is introduced as a factor of the integral and the solution is assumed as a power series in λ. It is shown that the coefficients are uniquely determined and that the series represents an integral function of λ, uniformly convergent with respect to x. Furthermore it is shown that this is the only finite solution. The solution of (1) is then obtained by integration and gives Volterra's theorem:

If $N(x, y)$ and $F(x)$ are differentiable with respect to x in the interval $(0a)$ and if $N(x, x) \neq 0$, and $F(0) = 0$, the equation (1) admits a unique solution which is finite and continuous in this interval.

The restrictive conditions of the preceding theorem are not necessary for an equation of the second kind. Proceeding with the consideration of the latter type, the formulas of Volterra for the iterated and reciprocal functions are obtained by applying Dirichlet's formula to the coefficients in the series obtained above.

Section IV of the first part explains the connection between

the equation of Volterra and linear differential equations. Every linear differential equation can be reduced to an equation of Volterra from which can be deduced immediately all the elementary properties of the solutions. It is interesting to note that by this process it is possible to reach the conception of a linear differential equation of infinite order. Under suitable conditions of convergence the existence and uniqueness of the solution follow.

The extension to equations involving several variables and to systems of equations offers no essential difficulties and the brief treatment merely indicates the procedure by the method of successive approximations.

The equation of Fredholm, which forms the subject of the second part of the book, differs from the equation of Volterra in that the upper limit of integration, as well as the lower limit, is constant. The problem is to find a function φ which satisfies the equation

$$\varphi(x) + \lambda \int_0^1 N(x, s)\varphi(s)ds = f(x). \tag{3}$$

The method of solution proposed by Fredholm is said to be considered by Darboux and Picard the most beautiful in analysis. In the exposition of this method the author has followed the works of Fredholm, Goursat, and Heywood.

Applying the method of successive approximations, the solution is obtained as a power series in λ, just as in the case of Volterra's equation. There is, however, an important difference. The solution of Fredholm's equation is not an integral function of λ, but is convergent if $|\lambda| < 1/N$, where N is the maximum value of $N(x, s)$ in the region considered. The solution can be written in the form

$$\varphi(x) = f(x) - \int_0^1 \Re(x, s, \lambda)f(s)ds,$$

where the reciprocal function (noyau resolvant)

$$\Re(x, y, \lambda) = N(x, y) - \lambda N_1(x, y) + \cdots + (-1)^p \lambda^p N_p(x, y) + \cdots.$$

The analytic character of the solution with respect to λ depends in the first place on that of $\Re$. The study of the solution is

then thrown upon the study of the reciprocal function. The second member of equation (3) plays no part in this consideration.

It is shown that the reciprocal function satisfies two integral equations, one of which is

$$(4) \qquad \Re(x, y, \lambda) = N(x, y) - \lambda \int_0^1 N(x, s)\Re(s, y, \lambda)ds.$$

The analogy with a system of algebraic equations suggests that the solution of (4) should be a meromorphic function of λ, in which the denominator does not depend upon x and y. Accordingly the solution is assumed in the form

$$(5) \quad \Re(x, y, \lambda) = \frac{A_0(x, y) + \lambda A_1(x, y) + \cdots + \lambda^p A_p(x, y) + \cdots}{1 + a_1\lambda + \cdots + a_p\lambda^p + \cdots} = \frac{D(x, y, \lambda)}{D(\lambda)}.$$

Substituting (5) in (4) and equating coefficients of powers of λ shows that the coefficients a_p may be chosen arbitrarily. If $a_p = 0$ the series found above is obtained. In order to get the formulas of Fredholm the relation

$$(6) \qquad pa_p = \int_0^1 A_p(s, s)ds$$

is assumed. It follows then that the coefficients A_p are uniquely determined and that the functions $D(x, y, \lambda)$ and $D(\lambda)$ are integral functions of λ. This method of exposition is very neat, but appears artificial because no reason is suggested for the choice of the relation (6). The function $D(\lambda)$ is called the determinant of the kernel N and the roots of $D(\lambda) = 0$ are the characteristic values.

The second chapter of part II is devoted to a more detailed study of the reciprocal function. The first two paragraphs deal with orthogonal and biorthogonal systems of functions, followed by a treatment of the kernel of the special form

$$N(x, y) = \sum_{p=1}^{n} \varphi_p(x)\psi_p(y).$$

Returning to the general case, it is shown that if λ_1 is a charac-

teristic value then $\lambda = \lambda_1$ is a pole of the reciprocal function. For $\lambda = \lambda_1$ the expression (5) is no longer valid. In general for $\lambda = \lambda_1$ the equation (3) does not admit a finite solution. To examine this case more in detail we consider the homogeneous equation, that is, suppose $f(x) \equiv 0$, and reach the second and third theorems of Fredholm:

II. For a characteristic value λ_1 of multiplicity n and rank (index) r, the homogeneous equation admits r linearly independent solutions called fundamental solutions. The associated equation (obtained by interchanging the variables in N) has exactly the same number of fundamental solutions.

III. The necessary and sufficient condition that the equation of Fredholm with second member $f(x)$ shall admit a solution for a characteristic value $\lambda = \lambda_1$ is that $f(x)$ shall be orthogonal to all the fundamental solutions relative to λ_1 of the associated equation.

Chapter III of the second part is entitled Special Kernels and considers principally symmetric and skew-symmetric kernels, and the development of functions in terms of fundamental functions, following the work of Hilbert and Schmidt.

The third part of the book treats of singular equations and non-linear equations. A linear integral equation is said to be singular (1) if one of the limits of integration is infinite, or (2) if the integrand becomes infinite for at least one point in the interval of integration, or (3) if, in the equation of Volterra of the first kind, $N(x, x) = 0$ for at least one point of the interval. This chapter of the theory of integral equations is not complete, and much is yet to be done towards obtaining results of a general character. The present theorems apply for the most part to special cases and show that remarkable analytic circumstances of the most diverse character may occur. For example, the characteristic values, in general, no longer form a discrete set, but may be distributed upon segments of the real axis everywhere dense, and the analytic nature of the solution in λ depends essentially on the second member of the equation.

The extensive bibliography is practically complete so far as the theory is concerned. No attempt is made to cite references to papers dealing with the applications.

Unfortunately the present edition of Lalesco's book contains many misprints which will be a source of some inconvenience to the reader.

The book by Heywood and Frechet contains three chapters. In the first is found a statement of certain problems which lead to an equation of Fredholm. They involve principally the determination of a harmonic function in a domain D of three dimensions, that is, a solution of the equation of Laplace

$$\Delta V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0,$$

which is analytic in D. Further conditions are necessary if D is infinite. The following functions are harmonic outside the attracting masses: (1) the potential of a space distribution; (2) the potential of a simple distribution on a surface, (3) the potential of a double distribution on a surface. The classic problems relative to harmonic functions are

1. *Dirichlet's Problem.*—The function V is subject to the condition of taking given values upon the surface S. $V_s = f(M)$, where M denotes a point of the surface.

2. *Neumann's Problem.*—In this case the values of the normal derivative upon S are given. $\partial V/\partial n_s = g(M)$.

3. *Problem of Heat.*—The expression $\partial V/\partial n + pV$ is given upon S, p being a function of M. The condition $q \cdot \partial V/\partial n + pV = h(M)$ reduces to the preceding if $q \neq 0$.

4. The term "mixed problem" is applied to a problem which one meets in hydromechanics and corresponds to $q = 0$ upon part of the surface and $p = 0$ upon the other part.

The problems of mathematical physics corresponding to the preceding occur in connection with the newtonian attraction and electrostatics, magnetism, hydrodynamics, elasticity, theory of heat, and acoustics. It is shown how these problems lead to integral equations of the Fredholm type.

The second chapter treats the theory of Fredholm's equation. The first solution is obtained by Neumann's method of successive substitutions and is made without the introduction of a parameter. Throughout this chapter the method of exposition and the terminology follow closely the corresponding work in Bôcher's Introduction to the Study of Integral Equations. The equation of Volterra is treated as a special case of the Fredholm equation.

In the third chapter the theory is applied to the solution of the problems stated in chapter I.

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AN ADVANCE IN THEORETICAL MECHANICS.

Théorie des Corps déformables. Par E. COSSERAT et F. COSSERAT. Paris, A. Hermann et Fils, 1909. vi+226 pp.

IN a recent review* of the French edition of the first half of Chwolson's monumental treatise on physics, we purposely and explicitly omitted comment upon two extended notes† of the Cosserats on the foundations of analytical mechanics other than first to state that, owing to the difficulty of the notes themselves and the lack of a general theoretical treatment of mechanics in Chwolson's volumes, these notes seemed out of place, and second to promise that these important contributions of the Cosserats should shortly be reviewed. The Théorie des Corps déformables is a reprint, with repagination, of the second of these notes, and may be obtained separately by those who may want it without the rest of Chwolson's highly valuable treatise.

The underlying idea of the Cosserats is to base analytical mechanics, in its most widely extended sense, not upon Newton's laws or Hamilton's principle, but upon what they call *euclidean action*, and to define the common concepts of mechanics in terms of this euclidean action; to develop a general theory which shall include as special cases newtonian mechanics and its various derivatives or extensions, of which there are so many, hydrodynamics, elasticity, and electromagnetism, and which shall extend in its generality to an infinity of possible non-newtonian systems of mechanics, of which at least one is now familiar to students of relativity. This ambitious task they have certainly accomplished, and from the favor in which their work seems to be received by such authorities as Appell, it is by no means impossible that Hamilton's principle, which up to the present has contained the most general and unifying theory of mechanics, may rapidly become replaced by the Cosserats' euclidean action.

The fundamental geometric element of their system is not the point, but the point carrying a system of rectangular axes, that is, the trirectangular triedral angle. It is clear that the

* This BULLETIN, volume 18, pp. 497–508, July, 1912.

† Chwolson, Traité de Physique, volume 1, pp. 236–273, and volume 2, pp. 953–1173.

point alone could not be sufficient; for an elastic filament differs from a geometric curve in the way in which a continuous series of trirectangular triedral angles differs from the locus of the vertices of the angles. Consider a function W of two neighboring positions of the triedral angle, that is, a function of the coordinates of the vertex, of the nine direction cosines of the edges of the angle, and of the first derivatives of these coordinates and direction cosines with respect to the time (in the case of a moving body) or with respect to the arc (in the case of the elastic filament).* Any such function W which has the property of being invariant under the transformations of the euclidean group is called the euclidean action; the quantity Wdt (or Wds_0, where ds_0 is the element of arc of an undeformed elastic filament) is called the euclidean action in the time interval dt (or in the element ds_0 of arc), and is likewise invariant under the euclidean group; the function W may contain explicitly the arc coordinate s_0 or (under appropriate conditions) the time t.

If the theory of the elastic surface is desired we have again to determine a function W of two neighboring positions of the triedral angle (that is, of the position of the vertex of the angle, of the direction cosines of the edges, of their first derivatives with respect to the coordinates ρ_1, ρ_2 of position on the surface, and of ρ_1, ρ_2 themselves), so that W shall be invariant under the euclidean group. The case of the dynamics of an elastic filament is formally identical when one of the coordinates ρ_1, ρ_2 becomes the arc and the other the time. And so on for more complicated cases. The condition that W shall be invariant specializes the form of W in a remarkable manner. In the first place the coordinates of the vertex of the angle cannot occur in W, and in the second place the direction cosines and the various first derivatives cannot occur except as implied in the "velocities" ξ, η, ζ or the "angular velocities" p, q, r associated with the triedral angle.†

The authors then consider the total action between the

* The analogy between the motion of an elastic medium of k dimensions and the equilibrium of an elastic medium of $k+1$ dimensions is, or should be, well known to all students of mechanics; it is this analogy which enables the authors to give a uniform treatment to dynamic and static problems of different natures.

† The terms "velocities" and "angular velocities" are included in quotation marks because in the statical problem they are not true velocities and angular velocities but the corresponding derivatives with respect to non-temporal coordinates such as ρ_1, ρ_2.

bounding conditions of the problem, for example

$$A = \int_{t_0}^{t_1} W dt, \quad A = \int_{A_0}^{B_0} W ds_0, \quad A = \int\int W dS_0,$$

and they employ the principle of varying action to define the fundamental quantities such as momentum, force, work, kinetic energy. The whole system of mechanics thus becomes one of nominal definition based upon the function W. The generality, uniformity, and precision of the treatment are striking, and this general statement of the method is intended to emphasize as much as possible these characteristics even at the expense of being vague. We shall now, therefore, discuss a particular application in a little detail for the sake of relieving the vagueness.

In the case of the dynamics of a particle the vertex of the triedral angle is alone significant and the edges must be ignored. The function W is then necessarily a function of the velocity v alone, provided we refer the particle to fixed, not moving, axes. The total action is

$$A = \int_{t_0}^{t_1} W(v) dt.$$

The variation of the action, after an integration by parts becomes

$$\delta A = [F\delta x + G\delta y + H\delta z]_{t_0}^{t_1} - \int_{t_0}^{t_1} (X\delta x + Y\delta y + Z\delta z) dt,$$

where

$$F = \frac{dW}{dv}\frac{1}{v}\frac{dx}{dt}, \quad G = \frac{dW}{dv}\frac{1}{v}\frac{dy}{dt}, \quad H = \frac{dW}{dv}\frac{1}{v}\frac{dz}{dt},$$

$$X = \frac{dF}{dt}, \quad Y = \frac{dG}{dt}, \quad Z = \frac{dH}{dt}.$$

Then F, G, H are by definition the components of momentum and X, Y, Z are by definition the component external forces. In like manner

$$X\delta x + Y\delta y + Z\delta z \quad \text{and} \quad F\delta x + G\delta y + H\delta z$$

are respectively the work done by the external force and the

work done by the momentum.* From these definitions we may prove that, as usual, the impulses of the forces and of their moments are respectively the change of the components of momentum and of moment of momentum.

The work done by the forces becomes at once

$$Xdx + Ydy + Zdz = d\left(v\frac{dW}{dv} - W\right),$$

an exact differential, and if we set

$$E = v\frac{dW}{dv} - W$$

and define E as kinetic energy, we have the familiar theorem as to the equality of the work done and the change of kinetic energy. We may define the mass of the particle in a number of ways dependent on the point of view we desire to adopt. Thus we have several masses:

1°, the quotient of the momentum by the velocity, called the maupertuisian mass, $1/v \cdot dW/dv$,

2°, the quotient of the action by half the square of the velocity, the hamiltonian mass, $2W/v^2$,

3°, the quotient of the kinetic energy by half the square of the velocity, the kinetic mass, $2E/v^2$.

Now if it be supposed that the mass is finite and the $W(v)$ is developable in powers of v, the development must begin with the second power,

$$W = \frac{m}{2}v^2 + (\,)\,v^3 + \cdots.$$

For sufficiently small values of v the terms of order higher than $\frac{1}{2}mv^2$ may be neglected. The system of dynamics founded upon the approximation $W = \frac{1}{2}mv^2$ is none other than the newtonian, and the three masses above defined reduce to the same constant m. Thus newtonian mechanics appears as the theory of slow motions, of motions infinitely near to the state of rest. The distinction between slow and fast motions is indeed analogous to that between infinitesimal and finite displacements in the theory of elasticity. From

* The authors use the term work in both cases despite the evident difference of physical dimensions, just as they use action for W or Wdt. This seems unfortunate.

the point of view of the principle of relativity, motion cannot be infinitely fast; those interested in these theories may discuss the mechanics founded on the assumption $W = m_0(1 - \sqrt{1 - v^2})$.

It would be useless here to follow in detail the three great sections of the text, namely, equilibrium of an elastic filament, equilibrium of an elastic surface and dynamics of an elastic filament, and equilibrium and motion of a continuous medium; they will be followed in detail by all earnest students of these topics. We prefer here to point out an advantage and a disadvantage of the Cosserats' system. In the main these are those associated with the transfer of any deductive-intuitional physical science to the corresponding formal-deductive mathematical discipline. The gain is in sureness, in freedom from constant doubtful appeal to intuition; a great variety of possible assumptions and corresponding cases may be discussed systematically and accurately from a given uniform point of view. Those who have taught the theory of elasticity will most appreciate this advantage. The loss is in the lessened training of that physical intuition, which is vital for the future success of the young physicist and which can be acquired only by practice in making such various plausible, but not demonstrated, assumptions as are frequent in the theory of elasticity. The simplest explanation of the world already subdued may in the last analysis be mathematically formal; but the subjugation of the regions not yet reached can in the first instance be accomplished only by the imagination.

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SHORTER NOTICES.

Opere Matematiche. By GIULIO CARLO DEI TOSCHI DI FAGNANO. Pubblicate sotto gli Auspici della Società Italiana per il Progresso delle Scienze. Per cura dei Professori Senatore Vito Volterra, Gino Loria, e Donisio Gambioli. Rome, 1912. 3 Vols. 40 lire.

IF we were to select from the great mathematicians of the world a half dozen whose works we might deem worthy of

new editions, or whose scattered monographs deserve to be brought together, it is hardly probable that the name of Fagnano would be in the list. Well known as he was, nevertheless he was hardly one of the world's great builders of science, nor was he one who laid deep its foundations or extended its applications into fields of great importance.

Born of a distinguished family that occupied the Castle of Fagnano for many generations, and that later settled in Senigallia, a family that had provided one pope and many distinguished citizens, the Conte Giulio Carlo early gave evidence of his noble breeding. At the age of ten he cultivated poetry, and soon after he published some philosophical verse on the resurrection. He studied in the Collegio Clementino at Rome, devoting his attention exclusively to letters, philosophy, and theology. Mathematics was not a favorite of his at first, but he seems to have been led to study the science through the philosophy of Leibniz, Wolf, Newton, and Malebranche.

It was in 1718 that Fagnano, then 36 years old, published his best-known memoir, the "Metodo per misurare la lemniscata," a contribution that at once attracted attention and was followed by numerous other essays, most of them appearing in the *Giornale de' Letterati d'Italia*. In 1743 an event happened which led to the collection and amplification of his memoirs, namely, the discovery of the insecurity of the dome of St. Peter's. Pope Benedict XIV, wearied of the discussions among Boscovich, Jacquier, Le Seur, and the Roman architects, and hearing from Monsignore Nicola Antonelli of Senigallia of the abilities of Count Fagnano, sent for him, and was so impressed with his powers that he ordered the publication of his works. This was in 1743, but it was not until 1750 that the Produzioni matematiche actually appeared, and then under circumstances not altogether pleasing with respect to the relations between the author and his patron.

The lemniscate, as is well known, was first described by the Bernoullis in 1694. To them Fagnano gives credit in the opening sentence of his Metodo. His work consisted chiefly in measuring the curve and in first bisecting its quadrant, and then dividing it into 2·2, 3·2, and 5·2 equal parts. It was because of this work that he felt justified in speaking of it as "mia curva," nor can we take exception to this claim as made in the investigations beginning in 1718. It was Euler, how-

ever, to whom the analytic theory is chiefly due (in volume 5 of the *Mémoires de St. Pétersbourg*).

An examination of the Produzioni shows that it is as an algebraist that Fagnano deserves chief recognition. To be sure he devotes a large amount of space to his Teoria generale delle proporzioni geometriche, so much space that few readers will be found who care to master it, but it is in his Applicazione dell' algoritmo nuovo that one finds displayed an interest in the field of mathematics in which Italy stood preeminent from the time of Ferro and Florido. Fagnano's treatment of equations, his ability to handle skilfully the complex number, and his contributions of a *nuova maniera* of handling cubic and quartic equations, are quite as noteworthy as his discoveries in the theory of the lemniscate.

Of the three volumes edited and published with such care by Professors Volterra, Loria, and Gambioli, the first contains Volume I of the Produzioni matematiche, devoted chiefly to geometric proportion and the " new algorism " applied to the treatment of equations. The second contains Volume II of the Produzioni, chiefly concerned with the theory of the triangle, special problems in the calculus, and the lemniscate. The third volume contains Fagnano's other scientific and polemic writings, a large number of his letters, and his biography.

Whether or not one feels that the standing of Fagnano justifies the republication of his memoirs and his *opus magnus* before those of other scientists whose works are out of print, he cannot deny the value of the labor undertaken by the Società Italiana per il Progresso delle Scienze, nor withhold the praise that is so justly due to Professors Volterra, Loria, and Gambioli.

DAVID EUGENE SMITH.

The Method of Archimedes Recently Discovered by Heiberg. A Supplement to the Works of Archimedes, 1897. Edited by Sir THOMAS L. HEATH, K.C.B., Sc.D., F.R.S., Sometime Fellow of Trinity College, Cambridge. Cambridge University Press, 1912. 51 pp. Two shillings and sixpence net.

IT is nearly four years since there appeared in *The Monist* Miss Robinson's translation from the German of the treatise on mechanics by Archimedes discovered by Heiberg in 1906. It was this reviewer's privilege to write a brief introduction to that translation, and all this material appeared in pamphlet

form a little later. It is therefore with a personal pleasure that the reviewer calls attention to the pamphlet prepared by Sir Thomas L. Heath as a supplement to the well-known edition of the works of Archimedes that appeared in 1897.

The pamphlet has several advantages over the one prepared in 1909. In the first place the introductory note is more complete, having been prepared with the added information given out by Heiberg in 1910 in Volume I of his new edition of the text of Archimedes. Again, it is prepared with the wealth of learning that only Sir Thomas Heath or Professor Heiberg could bring to such a task. And finally, the translation of the text is from the Greek instead of through the German, and has the advantage of the author's profound knowledge of the idioms to be found in the works of the Greek mathematicians.

The chief value of the work lies in the fact that it sets forth the method followed by the great Syracusan in making his discoveries in mechanics, and to the testimony that it bears to the fact that Democritus instead of Eudoxus should be credited with the discovery that the volume of a pyramid or a cone is one third of the volume of the corresponding prism or cylinder.

A word of commendation should also be given to the clear way in which the translation has been arranged upon the page, so that, as in the edition of 1897, the eye easily follows the proof.

DAVID EUGENE SMITH.

Didaktik des mathematischen Unterrichts. Von Dr. ALOIS HÖFLER, O.O. Professor an der Universität Wien. Leipzig, B. G. Teubner, 1910. Mit zwei Tafeln und 147 Figuren im Text. xviii+509 pp. 12 Marks.

THE present day in the teaching of secondary mathematics, and in a less degree of all mathematics, is characterized by a spirit of unrest in every progressive country in the world. No one person is responsible for this state of affairs, and not very many leading names are connected with it. It is not a campaign carried on by field marshals in education or in mathematics; it is rather a mass movement without other leader than the Zeitgeist; it is democracy asserting itself against the old aristocracy of learning; it is often merely an effort to have things different, with no well-defined plan of having them better. This desire for change shows itself

in various aspects. In England it has recently been directed to the elimination of Euclid, with the result that matters mathematical are temporarily in a condition that can hardly be satisfactory to anyone. In America it often assumes the form of petty mathematics, ill arranged courses, or the hope that we may no longer have any mathematics whatever required in the high school. In France it shows itself in the effort to replace Legendre's geometry of congruence by a geometry of motion that seems at present more abstract and ill arranged than anything that has preceded it. And so, in all countries, we find this longing for something better, but often with the result that something worse appears. Now there is no doubt that, with the admission of the mass of boys and girls to high-school training, in place of a selected lot as in the past, we have to lower, temporarily at least, the general standard. This is sometimes interpreted to mean that the heart must be cut right out of secondary mathematics and that we are to have merely a little weak, diluted algebra and geometric drawing, with no serious work requiring sustained effort and logical reasoning. In particular it is often asserted that the Realanstalten, which appear with us as technical high schools and vocational schools of one kind or another, should have only mathematics that is immediately practical, the idea of the potentially practical being lost in the desire for the present need. Some go so far as to assert that mathematics should not appear as a science at all, but that when a real problem arises its solution should be effected and no other problems should be given.

This preliminary statement, long as it is, is necessary to an understanding of the need for a book like this by Professor Höfler. Written as the first one of a series of didactic manuals "für den realistischen Unterricht an höheren Schulen," it aims directly at the problem of improving the teaching of mathematics, and of giving to the science a firm basis, in the modern type of school. It is written with an earnestness of purpose that is gratifying to every teacher of the subject and that should be gratifying to everyone who has to do with the education of the youth, but that will not be at all appreciated by the type of mind that wishes to destroy instead of construct.

The work is divided into three parts. Of these the first has to do with the purposes and methods of mathematical instruction, and with modern questions of values and of topics

to be considered. For example, the question of the function concept, of "functionally thinking" in mathematics, is considered at much length. This question, of which we shall be hearing a great deal in the next ten years in the secondary schools of America, was first raised for such schools in France some time ago, chiefly by the late Professor J. Tannery, and was then taken up by Professor Klein and given a definite standing in Germany and Austria, whence it is making its way into other countries. What this innovation for our secondary schools means, why it is undertaken, and what purposes are supposed to be accomplished, are considered in this work, and should be carefully considered by all teachers before we tend to go to the extreme that usually characterizes our schools whenever a novelty is suggested.

The second part relates to the course of study, the characteristics of the successive stages of the pupil's progress, and the treatment of distinctive topics. It is always helpful, under the common accusation of superficial education that is advanced against us, to see with what thoroughness the Teutonic schools do their work. Hence a teacher of high-school mathematics can hardly find better reading in his field of activity than Professor Höfler's sections on topics like irrational and imaginary numbers, algebraic and transcendent numbers, the relation of mathematics to physics, and the introduction to the calculus.

The third part considers the relation of psychology, logic, and philosophy to mathematics. The significance of number, of space, of intuition in mathematical teaching, and of fundamental concepts, the relation of logic to mathematical proof, the interesting and not sufficiently appreciated question of "mathematical sophisms," and a review of mathematical values, are the leading features of the concluding part of the work.

We have great need of books of this type, and of those that have thus far appeared this is, to say the least, one of the best. The author does not show that familiarity with the literature of other languages that might reasonably be expected from one in his position and with his scholarship, or from a writer of such a treatise; but as a representative of the best Teutonic thought the work will long be recognized as a standard authority.

DAVID EUGENE SMITH.

Der Wert der Wissenschaft. Von HENRI POINCARÉ. Uebertragen von E. WEBER, mit Anmerkungen und Zusätzen von H. WEBER. Zweite Auflage mit einem Vorwort des Verfassers. Leipzig, Teubner, 1910. 8vo. viii + 251 pages. 3 Marks.

THIS edition of the second volume in Poincaré's series of three volumes dealing with the fundamental concepts of science appeared after the book had reached its fourteenth thousand in the French editions. This fact indicates the widespread interest which it inspired. The masterly way in which the real significance of scientific facts and theories is brought out, and that which is permanent and abiding in the shifting lights of the new discoveries of science made evident, appeals to every philosophic mind.

This particular edition is enriched with a preface by Poincaré, in which he considers the service that industrial science renders to pure science. After speaking of the amazement of the multitude that the truth of today in science becomes so easily the error on tomorrow, and their consequent belief that the discoveries of science are after all of less significance than we suppose, he continues thus:

"And I am not speaking here of those abstract truths that have become so general that they have lost all precise significance; truths spelled in capitals and a source of wonder, but of whose meaning we can say nothing. The permanent truths of science are the facts, not only the crude facts but also the true relations between the facts; what changes is the language in which the facts are expressed; the mode of expression changes because on the old facts falls the light reflected by the new, which are discovered every instant and which must be expressed as well as one can, not only in their own light but in the significance of the illumination from many sources.

"Happily science is needed for its applications, and this fact silences the sceptic. If he desires to use some new discovery and convinces himself that it is a success, he must indeed admit that there is more there than an idle dream. Thus we perceive the blessing in the development of industry.

"I do not wish to say that science is made for its application, far from it; one must love it for its own sake; but the recognition of its uses protects us from the sceptic.

"And then too the enemies of science produce another argument: they observe that many discoveries are made by

men of no great education; that the learned receive these discoveries at first with shrugs of the shoulders, and prove that there is nothing in them; but after the discoverer has persevered, and even come to success through a kind of contempt for science, then the learned demonstrate the possibility of success.

"And it is often partially true; bold undertakings are frequently due to those who are free from dizziness, and to prevent dizziness one must not see too clearly. Of these adventurers only the successful are counted, not those who break their necks.

"Not the less true is it that modern industrial development, considered as a whole, would have been impossible but for the advance of science. The unlearned live daily in an environment created by science, and unconsciously receive the benefit. It is science that gives a form to their dreams, which in other centuries would have been very different. Many bring to their applications ideas of scientific origin, but which their discoverer looked upon as only the mind's play, and impractical, because he foresaw a thousand difficulties. Every inventor has had predecessors whose great merit was the fact that they did not halt at difficulties, which they did not perceive simultaneously, but conquered one at a time."

There are thirty-seven pages of notes by H. Weber. These consist of explanatory remarks, historical notes on the original text, and some philosophical considerations. They will be particularly useful to the general reader. The last note closes with a quotation ascribed to Novalis:

"The life of the Gods is mathematics. All divine messengers must be mathematicians. Pure mathematics is religion. Mathematicians are the only fortunate ones. The mathematician is naturally an enthusiast. Without enthusiasm no mathematics."

JAMES BYRNIE SHAW.

Spezielle Flächen und Theorie der Strahlensysteme. Von Rektor Dr. V. KOMMERELL und Prof. Dr. K. KOMMERELL. Sammlung Schubert LXII. Leipzig, Göschen, 1911. vi+171 pages.

DURING the preparation of the second edition of the authors' Allgemeine Theorie der Raumkurven und Flächen,* it was

* Sammlung Schubert, XXIX and XLIV. First edition 1903; second edition 1910–11.

found that, in the process of revising and adding desirable new material, the second volume was becoming too large. At the suggestion of the publishers this volume was divided and the second part appears as a separate book. In comparing this new edition with the old, it is to be noted that the general character of the text has not been changed. It is still a first introduction to differential geometry, aiming to present the fundamental facts and principles in a simple and concise manner. Brevity, however, is no longer a striking feature, for there are 531 pages in the three volumes of the new edition.

The American student beginning the study of differential geometry now will probably not use this second edition so frequently as his predecessor employed the first one. When the latter appeared, four years after the German edition of Bianchi's book, there was no text in the English language. But now we have Professor Eisenhart's excellent work.

In the present volume the first 90 pages are devoted to the "special surfaces," including W-surfaces, minimal surfaces, surfaces of constant curvature, ruled surfaces, and triply orthogonal systems of surfaces. Here there are various minor alterations and additions, especially under the first three headings. But the greatest changes are to be found in the next 60 pages, which deal with rectilinear congruences. The sections on isotropic congruences are perhaps the most noteworthy, not only because of the fact that the treatment here is fuller than in most of the texts; but, also, because these ten pages contain material from a recent article* by one of the authors. Of especial interest are the remarkably simple formulæ for the middle surface of the most general isotropic congruence.

The book is concluded by a collection of thirty-one problems.

E. B. COWLEY.

Einführung in die Theorie der partiellen Differentialgleichungen. Von Dr. J. HORN, Professor an der Technischen Hochschule zu Darmstadt. Leipzig, Göschen, 1910. vii+360 pp.

THIS work may be considered as the third volume of the course in differential equations published in the "Sammlung Schubert." The first volume is the well known work by Schlesinger, Einführung in die Theorie der Differentialgleich-

* *Math. Annalen*, vol. 70, pp. 143–160.

ungen mit einer unabhängigen Variabeln.* The second volume is Gewöhnliche Differentialgleichungen beliebiger Ordnung, by the author of the book under review. The task which Dr. Horn seems to have set himself was to produce in one small volume a course in partial differential equations which was to be readable for the mid-course student, rigorous in treatment and such as to bring out a number of points of view of both the older and the more modern theory. In other words the book was to fill up the gap between the rather meager discussion of partial differential equations in the last chapters of treatises on analysis and the works on special topics or special methods of treatment.

After the first chapter, which takes up existence proofs for linear partial differential equations of the first order and with n independent variables, the author restricts himself to equations of the first and second orders with two independent variables. In the chapter on equations of the first order he gives a special existence proof and then takes up Cauchy's method of integration by means of the characteristics, using Darboux's geometrical language. The methods of Lagrange and Monge are used in the discussion of the complete integral. In the third chapter, after giving Goursat's existence proof for partial differential equations of the second order and discussing in general the characteristics and their relation to the integral surfaces, the particular equation of Monge and Ampère

$$Hr + 2Ks + Lt + M + N(rt - s^2) = 0$$

(H, K, L, M, N functions of x, y, z, p, q)

is taken up with the linear equation of the second order as a special case. Laplace's method of solution is also used to introduce the subject of invariants. The fourth chapter is devoted to hyperbolic partial differential equations of the second order, in particular to the discussion of the integral by means of Green's theorem. The methods of Riemann are also briefly studied.

In the fifth chapter begins the characteristic half of the book, which up to this point differs only in conciseness and arrangement from other texts, say parts of the fifth and sixth volumes of Forsyth. Before introducing the reader to the

* See review in the BULLETIN, vol. 8, p. 168.

elliptic partial differential equation, the author includes a good introduction to the integral equation according to Fredholm. The chapter is clear and not too brief for the reader for whom the book is intended, though the following chapter on boundary problems in ordinary linear differential equations of the second order might have been shortened with profit in a work of this type and title. One quarter of the book is devoted to the introduction to integral equations. In chapter seven the results of the two previous chapters are applied to particular equations of the elliptic type. Properties of the solutions of

$$\Delta u = 0,$$

in particular two boundary value problems, are treated at some length, following Fredholm and Hilbert in treatment and notation. A few pages are devoted to special points connected with the solutions of

$$\Delta u + 2\pi\varphi(x, y) = 0, \quad \Delta u + \lambda u = 0,$$

$$\Delta u + \lambda k(x, y)u = 0 \ (k > 0).$$

The volume closes with a short chapter on some partial differential equations of physics.

The book is clear and logical. Generalities are illustrated by well-chosen special examples. After deciding upon the content the author keeps to the point and does not forget the student for whom he is writing. As to the content, of course there will be differences of opinion as to the choice of topics from such a wide field. For example it would not have been a difficult task to give some notion of Lie's methods without an increase in size. This volume is well worthy of a place in a series which includes Schlesinger's little work.

A. R. CRATHORNE.

Lehrbuch der Differentialgleichungen. Von A. R. FORSYTH. (Mit den Auflösungen der Aufgaben von HERMANN MASER.) Zweite autorisierte Auflage, nach der dritten des englischen Originals besorgt und mit einem Anhang von Zusätzen versehen von WALTHER JACOBSTHAL. Braunschweig, Vieweg und Sohn, 1912. xxii + 921 pp.

FOR many years Forsyth's Treatise on Differential Equations has held a place of importance among physicists and

other workers who have much to do with the practical problem of solving differential equations. The value of the book in this respect is due to its fullness of practical methods of integration and its great number of well-chosen examples. The purpose of the second German edition is to extend the range of subjects beyond those embraced in the third English edition so as to make the book more useful to students of pure mathematics. Nevertheless no use is made of function-theoretic considerations; and the discussion is confined almost entirely to real variables.

For convenience of review the book may be divided into three parts. The first (pages 1–526) contains a translation of the third English edition; the second (pages 527–664) contains the new matter supplied to the German edition by the translator; while the third (pages 665–912) contains solutions of exercises. There is added an author index and a subject index.

The third part of the book contains Maser's solutions of the exercises found in the second English and in the first German edition, with numerous corrections made by the translator. No solutions of the additional problems contained in the third English and in the present German edition are given.

The first part of the book follows the third English edition. Throughout the treatment of ordinary equations there are many footnotes referring to the discussion given in the Zusätze in the second part, where additions to and sometimes criticisms of the statements in the text are to be found. No such additional remarks are adjoined to the treatment of partial equations. In at least one important case it seems desirable that such should have been done. The classification of integrals of partial differential equations (§§ 177–183) is not satisfactory. The classification is arrived at by means of a formal process of elimination and there is no inquiry as to the range of validity of the formal process. The resulting classification of integrals into three kinds is incomplete; there are integrals which do not come in any of these classes. For a discussion of this matter see Forsyth's Theory of Differential Equations, volume V, chapter V.

There remains yet for consideration the part of the book which contains new matter; and of this we shall speak in more detail. Only the principal additions will be considered, numerous shorter notes not being mentioned at all. In this part attention is confined entirely to ordinary equations.

Existence theorems are given (pages 529–541) for a single equation of the first order, for n equations of the first order, and for a single equation of the nth order. The argument is presented in such form as to be intelligible to one who has no further acquaintance with functions than is obtained in a good course on integral calculus. The method is that of Cauchy and Lipschitz. Thus we have a satisfactory treatment for a student's first introduction to the rigor of an existence theorem.

The short discussion of algebraic solutions and addition theorems (pages 545–550) cannot fail to be interesting and instructive to the student. The systematic treatment of the theory of integrating factors (pages 551–556) is also a welcome addition.

The general discussion of singular solutions given in the English edition (and reproduced in the present one) is somewhat too difficult for the beginner. Consequently, a more elementary treatment of this subject is given (pages 556–565). There can be no doubt that this addition is a desirable one.

An interesting treatment of the formal problem of integrating by series is given on pages 573–597. The discussion is limited to linear homogeneous equations. The only problem considered is that of the formal determination of power series which formally satisfy the equation. All convergence proofs are omitted. The general problem of finding the power series expansions which formally satisfy the equation is worked out in a way which is very satisfactory from the point of view of the beginner. The treatment is not entirely complete, some exceptional cases being omitted, as for instance, when the roots of the indicial equation differ by an integer. But when the equation is of the second order the treatment is made complete.

The important particular case when the recursion formula for determination of the coefficients of the series expansion has only two terms is given a special detailed treatment (pages 589–597). The importance of this case, especially in a student's early study of integration by series and in applications to equations of physics, will make this section a welcome one to many teachers. The matter is well presented and the results are stated in such form as to be easy of reference (as is indeed the case with all the more important results contained in the Zusätze).

The application (pages 597–617) of the theory of integration

by series to the Legendre, Bessel, and hypergeometric equations will be useful to the student.

The last of the important Zusätze (see pages 637–663) is devoted to an exposition of the theory of systems of simultaneous differential equations of the first order.

Having thus passed in quick review the contents of the book, it is now apparent that the translator has certainly accomplished his purpose of making it more useful to the student of pure mathematics. There remains the question whether there are not other additions which would have been desirable in accomplishing this purpose. There is at least one which, in the reviewer's opinion, should have been inserted.

In connection with the theory of formal integration by series it would have been easy to insert a proof of the convergence in general of the series for the case of second order equations; and one can but regret that this was not done. Such a treatment would have required only a half-dozen pages; and it would have added greatly to the value of this already valuable section. The translator's reason for omitting it is obvious; he was making no use of function-theoretic considerations, and such a proof would have required the introduction of these notions. But the great value to the student of having at hand this proof, for the relatively simple case of second order equations, seems to be more than a justification for departing in this case from the general plan of the book; it seems indeed to be a demand for it.

On the whole the translator has rendered a distinct service to beginners in the modern theory of differential equations. The Zusätze which he has inserted in this volume have to do with well-selected topics and the treatment is for the most part very satisfactory. The careful arrangement of material and the numerous and convenient cross-references given throughout the Zusätze and the Auflösungen are especially to be commended as contributing to the reader's comfort. The usefulness of such mechanical conveniences is often overlooked by authors. R. D. CARMICHAEL.

Calcul des Probabilités. Par H. POINCARÉ. Rédaction de A. QUIQUET. Deuxième edition, revue et augmentée. Paris, Gauthier-Villars, 1912. 335 pp.

THE first edition of Poincaré's Calcul des Probabilités was published in 1896; thus it has been before the public for sixteen

years. Therefore in this notice it will be necessary to indicate only the changes (not many in number) which have been made in the second edition. These are of two kinds: changes in content, and changes in arrangement and printing.

The principal changes in content consist of two additions. There is an introduction taken from the chapter entitled Le hasard in Poincaré's Science et Méthode. It has to do with the philosophical considerations connected with the possibility of a mathematical theory of probability. There is a fresh chapter at the close of the book dealing with a number of miscellaneous questions. Besides this there are some rearrangements of old matter and additions of new matter throughout the book; but in no cases do these changes seem to be of sufficient importance to require separate consideration.

In the first edition the material was grouped by lectures and not by topics, and no page headings were given to indicate the nature of the contents at any place. On this account the book was inconvenient for purposes of reference. In the second edition there is an arrangement of the matter by topics into chapters and page headings are given to indicate the chapter to which any page belongs. This adds greatly to the reader's comfort and will increase the usefulness of the book.

Concerning a work of Poincaré's, one scarcely needs to add that it is interesting and valuable to the student of the subject with which it deals.

R. D. CARMICHAEL.

The Dynamical Theory of Sound. By HORACE LAMB. New York, Longmans, Green and Co. (London, Edward Arnold), 1910. viii + 303 pp.

To the two hundred or more foreign mathematicians gathered at Cambridge last summer the atmosphere of the Congress may well have appeared somewhat foreign; for prominent among the "home talent" were Sir George Darwin, the president, Lord Rayleigh, the honorary president, Sir J. J. Thomson, Sir Joseph Larmor, M.P., and Professor E. W. Brown, three of the lecturers, all in the front rank of mathematicians in the Cambridge sense, but elsewhere ranked rather as physicists or astronomers. Indeed, although Cambridge has been and still is graced by the presence of eminent pure mathematicians, there is no more striking phenomenon in university history, no more persistent and justified tradition,

than the preeminence of Cambridge as *the* school of the dynamical explanation of the universe, foremost from Newton to Maxwell and on to the present day. Among the treatises, standards for the world, and issue of this school, are Lord Rayleigh's Sound and Lamb's Hydrodynamics; and now Lamb offers us out of his mature experience an elementary Sound, a sort of lesser Rayleigh, written in a delightful style and valuable both for itself and as an introduction to the greater work.

Recently we have had occasion to remark that geometrical optics has fallen between the mathematician and physicist into the hands of the optical engineer.* If we may judge by the pamphlets circulated by our university departments, sound has likewise, and probably for similar reasons, been abandoned by mathematician and physicist. Indeed one such pamphlet makes bold to state that sound is not the subject of any course in physics, but in some of its important aspects is treated in a certain course in mathematics. Needless to say, this course is on harmonic analysis and the aspects of the theory of sound therein treated are merely those connected with the determination of solutions of certain differential equations subject to particular boundary conditions. These problems are adequately treated by Lamb, not only from the mathematical, but from the dynamical and physical viewpoints. There are, however, numerous other phenomena of sound, sometimes connected with common and simple physical instruments, which the author discusses, and which he must sketch more from the physical side, less from the mathematical, because the complete mathematical solution offers too great difficulties in analysis.† Thus a considerable part of the work assumes a semi-descriptive tone.

Mathematicians are offered a number of interesting problems in the theory of sound somewhat more advanced than elementary harmonic analysis. For example, there is the theory of finite waves, where the differential equations are no longer linear. Riemann, Hugoniot, Hadamard ‡ is the sequence of names which should be mentioned in this connection. Lamb merely discusses the matter briefly with a reference to

* This BULLETIN, November, 1912, p. 74.

† The megaphone, for instance, is a simple object with familiar effects, but its mathematical theory is not by any means simple.

‡ See a review of Hadamard's Théorie des Ondes, this BULLETIN, vol. 10, pp. 305–317.

Riemann. Perhaps, however, the topic now most likely to draw mathematicians back to the abandoned theory of sound is the theory of integral equations and its applications to the integration of differential equations of physics. But unfortunately for the practical man the series which arise in Fredholm's solutions, though very rapidly convergent, are extremely difficult to compute, owing to the complexity of each term. It may be, therefore, that for special problems apart from existence theorems we may still be forced to use something more akin to harmonic analysis, perhaps the method of Ritz.* These matters the author very properly omits from this elementary treatise.

As for matters of detail, the book, after a short introduction, takes up the study of vibrations. The pace is moderate, passing successively the simple pendulum, the general system of one degree of freedom, forced vibrations, resonance, friction and damping, systems of several degrees of freedom, and the transition to continuous systems. The principle that the periods of the normal modes are "stationary" is not overlooked,—nor is the principle of reciprocity. Although both these principles are mathematical, they are unfortunately omitted from most treatises which are not primarily interested in the physical foundation of the subject. Chapter II is on the vibrations of strings, and leads to the study of Fourier's theorem (Chapter III). This is followed by a chapter on the vibrations of bars, and one on membranes and plates. These developments have filled about one half of the volume. Apart from the derivations of the differential equations, the considerations of energy, and the discussions of the various limitations which actual physical conditions may impose, the work is such as might well be found in a mathematical course on harmonic analysis.

In passing it should be observed that the page of this book has a very pleasing and restful appearance; it is neither too large nor too small, too open not too closely packed. The typography, though excellent, could be improved in two particulars: 1°, by the use of mortised integral signs, wherever a lower limit has to be set; 2°, by raising the periods and commas when occurring immediately after the central line of

* Reference may be made to the illuminating developments and comments of Poincaré, Leçons de Mécanique céleste, tome 3: Théorie des Marées, chap. 10.

a fraction. As this latter improvement, however, is unfamiliar in our BULLETIN and *Transactions*,* we regret throwing the stone!

In the sixth chapter, after discussing the elasticity of gases, the velocity and energy of plane sound waves, reflection, and the vibrations of a column of air, Lamb takes up waves of finite amplitude and the possibility of the propagation of a wave of discontinuity. He gets far enough to run into the difficulty, first signalized by Lord Rayleigh, in regard to the conservation of energy; and then, instead of discussing Hugoniot's law of dynamic adiabaticity, he remarks that obviously no complete theory of waves of discontinuity can be attempted without some reference to viscosity and thermal conduction, and he therefore proceeds to treat these two subjects. His remark is both right and wrong; it is wrong in stating that no complete theory of waves of discontinuity can be attempted without some reference to viscosity, for this is precisely what Hugoniot and his follower Hadamard have attempted and accomplished with results well in accord with Vieille's experiments; it is right because there is no à priori reason for disregarding viscosity, especially in the case of waves of discontinuity. The chapter ends with the treatment of the damping of waves in narrow tubes and crevices.

The author is now ready for the general theory of sound waves (Chapter VII), the general equations of fluid motion, specialized for sound, spherical waves, waves resulting from a given initial disturbance, point sources, reflection, refraction due to temperature gradients, and refraction by wind. Chapter VIII continues with spherical waves and point sources, and applies them to the waves given off by a vibrating sphere or other solids. The scattering of sound by an obstacle and its transmission through an aperture (diffraction) are briefly discussed. Only the theory of pipes and resonators (Chapter IX) and some account of physiological acoustics (Chapter X) is needed to round out and complete the work.

It should be evident that the author has exhibited excellent taste and balance in his selection and treatment of topics, that he has accomplished just what he intended, and that it was worth accomplishing,—a text that by easy stages fits the reader for the more advanced treatises and, we may add, a

* So far as we recall the only people who know how to set commas after fractions are Ginn and Co. and Gauthier-Villars; others drop them too low so that they either look lost or look like accents to the denominator.

text that by its graceful composition can hardly fail to lure the reader on to the further study of the subject. All this would have been predicted in advance of reading the Sound by anybody at all familiar with the author's Hydrodynamics.

EDWIN BIDWELL WILSON.

NOTES.

THE opening (January number) of volume 14 of the *Transactions of the American Mathematical Society* contains the following papers: "The triad systems of thirteen letters," by F. N. COLE; "Triple system sas transformations, and their paths among triads," by H. S. WHITE; "Proof of Poincaré's geometric theorem," by G. D. BIRKHOFF; "On the existence of loci with given singularities," by S. LEFSCHETZ; "Singular multiple integrals, with applications to series," by B. H. CAMP; "Decomposition of an n-space by a polyhedron," by OSWALD VEBLEN; "On convergence factors in double series and the double Fourier series," by C. N. MOORE; "Algebraic surfaces invariant under an infinite discontinuous group of birational transformations. Second paper," by VIRGIL SNYDER; "Note on Van Vleck's non-measurable sets," by N. J. LENNES; "Some asymptotic expressions in the theory of numbers," by T. H. GRONWALL; "Determination of the finite quaternary linear groups," by H. H. MITCHELL; "On the character of a transformation in the neighborhood of a point where its Jacobian vanishes," by L. S. DEDERICK.

THE December number (volume 14, number 2) of the *Annals of Mathematics* contains the following papers: "Two theorems on conics," by S. LEFSCHETZ; "A new type of solution of Laplace's equation," by H. BATEMAN; "Involutoric circular transformations as a particular case of the Steinerian transformation and their invariant nets of cubics," by A. EMCH; "On analytic functions of constant modulus on a given contour," by T. H. GRONWALL; "Necessary and sufficient conditions for the interchange of limit and summation in the case of sequences of infinite series of a certain type," by T. H. HILDEBRANDT; "A simple proof of a fundamental theorem in the theory of integral equations," by MAXIME BÔCHER; "An application of modular equations in analysis situs," by O. VEBLEN.

DURING the year 1912 the American Academy of Arts and Sciences published in its *Proceedings* (volumes 47 and 48) the following papers wholly or largely mathematical: "On an electromagnetic theory of gravitation," by D. L. WEBSTER; "An algebra of plane projective geometry," by H. B. PHILLIPS and C. L. E. MOORE; "A theory of linear distance and angle," by H. B. PHILLIPS and C. L. E. MOORE; "The impedance of telephone receivers as affected by the motion of their diaphragms," by A. E. KENNELLY and G. W. PIERCE; "The space-time manifold of relativity. The non-euclidean geometry of mechanics and electromagnetics," by E. B. WILSON and G. N. LEWIS; "On the existence and properties of the ether," by D. L. WEBSTER. These make a total of about 300 pages. Each article is published separately and can be obtained by purchase direct from the Academy, 28 Newbury Street, Boston, if authors are unable to supply copies.

AT the Cleveland meeting of the American association for the advancement of science Dr. FRANK SCHLESINGER, of Allegheny Observatory, was elected vice-president and Professor F. R. MOULTON secretary of Section A. Professor J. C. FIELDS was elected member of the sectional committee and Professor T. M. FOCKE member of the general committee. The next meeting of the Association will be held at Atlanta, Ga., in convocation week under the presidency of Professor E. B. WILSON, of Columbia University.

THE following papers were presented at the December meeting of the London mathematical society: "A connection between the functions of Hermite and Jacobi," by H. E. J. CURZON; "The equations of the theory of electrons transformed relative to a system in accelerated motion," by H. R. HASSÉ; "The convergence of series of orthogonal functions," by E. W. HOBSON; "Derivatives and their primitive functions" and "Functions and their associated sets of points," by W. H. YOUNG; "Mersenne's primes," by J. MCDONNELL; "The Diophantine equation $y^2 = x^3 + k$," by L. J. MORDELL.

WITH the support of the Christiania and Leipzig Academies, the firm of B. G. Teubner contemplates the publication by subscription of the Collected Works of SOPHUS LIE, edited by FRIEDRICH ENGEL. It is proposed to issue the Works in seven large octavo volumes, averaging about 600 pages, at a

total cost of about 160 Marks (single signatures of 16 pages at 60 Pf.). The plan is conditioned on the receipt, before April 1, 1913, of a sufficient number of subscriptions to justify the great expense involved. In view of the importance of Lie's work and ideas and of the fact that some of his papers are not easily accessible, it is hoped that this proposed monument to his genius may receive the necessary support. Subscriptions should be sent to B. G. Teubner, Leipzig. Upon the appearance of the first volume, the price will be considerably increased.

THE following changes have been announced in the faculty of science of the University of Paris: The chair of physical astronomy has been renamed the "chair of astronomy"; the chair of mathematical astronomy and celestial mechanics has been renamed the "chair of analytic and celestial mechanics" and Professor P. APPELL has been appointed to it; Professor P. PAINLEVÉ will occupy the chair of rational mechanics vacated by Professor Appell; the chair in general mathematics, vacated by Professor Painlevé, will not be filled for the present.

PROFESSOR G. MITTAG-LEFFLER, of the University of Stockholm, and Professor H. A. SCHWARZ, of the University of Berlin, have been elected corresponding members of the Munich academy of science.

PROFESSOR J. HADAMARD, of the Collège de France, has been elected a member of the Paris academy of science in succession to the late Professor H. POINCARÉ.

PROFESSOR F. ENRIQUES, of the University of Bologna, has been elected member and Professor E. PICARD, of the University of Paris, foreign associate of the Italian Society of Sciences (the XL).

PROFESSOR G. HAMEL, of the German technical school at Brünn, has been appointed professor of mathematics in the technical school at Aachen.

DR. TH. VON KARMAN, of the University of Göttingen, has been appointed professor of mathematics in the forestry school at Schemnitz (Hungary).

MR. G. RUHM has been appointed professor of mathematics in the Agricultural College at Bonn-Poppelsdorf.

DR. L. CRELIER, of the University of Bern, has been promoted to an associate professorship in geometry.

MR. E. S. PALMER has been appointed professor of mathematics in Rollins College, Winter Park, Fla.

AT the University of Manitoba, Mr. L. A. H. WARREN has been promoted to an assistant professorship in mathematics.

SIR G. H. DARWIN, Plumian professor of astronomy and experimental philosophy in the University of Cambridge, died December 7 at the age of sixty-seven years.

PROFESSOR WILHELM FIEDLER died November 19, 1912 at the age of 84 years. Professor Fiedler was connected with the technical school of Zurich for 40 years, retiring in 1907.

THE death is announced of Dr. R. SCHIMMACK, of the University of Göttingen, at the age of 31 years.

PROFESSOR WILLIAM J. VAUGHN, head of the departments of mathematics and astronomy at Vanderbilt University, died December 16, at the age of seventy-eight years. Professor Vaughn graduated from the University of Alabama in 1857 and was instructor and professor in that institution from 1860 to 1865 and again from 1878 to 1882. At Vanderbilt he held the chair of mathematics since 1882 and that of astronomy since 1895. He had been a member of the American Mathematical Society since 1900.

CATALOGUE of second hand mathematical books: R. Friedländer und Sohn, Karlstrasse 11, Berlin, catalogue 480, history and older authors to the time of Euler, 22 pp.— A. Hermann et Fils, 6 Rue de la Sorbonne, Paris, catalogue 118, 4465 titles.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Bes (K.). Uit de Theorie der algebraische Vergelijkingen. Haarlem, 1911. 8vo. 166 pp. M. 3.00

Beza (W.). Hemireguläre Polygone. (Progr.) Krems, 1912. 8vo. 5 pp.

Borel (E.). See Notice.

Breggen (J. van der). Vademecum d. wiskunde. Zutphen, 1911. M. 1.00

Bricard (R.). See Duporcq (E.).

Carslaw (H. S.). Introduction to the infinitesimal calculus. 2d edition. London, 1912. 8vo. Cloth. 4s.

Crepcoeur (A. J. M.). Cours d'analyse infinitésimale. Partie III: Calcul des probabilités ou théorie analytique du hasard. 2e édition. Anvers, 1911. 8vo. 253 pp. Fr. 7.50

Czuber (E.). Vorlesungen über Differential- und Integralrechnung. 2ter Band. 3te, sorgfältig durchgesehene Auflage. Leipzig, Teubner, 1912. 8vo. 10+590 pp. Cloth. M. 12.00

Duffek (J.). Beweis des Fermatschen Satzes. Mähr.-Trübau, Nowotny, 1912. 8vo. 15 pp. M. 0.50

Duporcq (E.). Premiers principes de géométrie moderne. 2e édition, revue et augmentée par R. Bricard. Paris, 1912. 8vo. 8+174 pp. Fr. 4.00

Ender (A.). Die konformen Raumtransformationen. 2ter Teil. Waidhofen a. d. Ybbs, 1912. 8vo. 24 pp.

Erb (T.). Erfüllung linearer Differenzen-Gleichungen durch Potenzreihen. (Progr.) Pirmasens, 1912. 8vo. 19 pp.

Falisse (V.). Cours de géométrie analytique plane. 7e édition, revue et augmenteé par A. Gob. Bruxelles, Lebègue, 1912. 6+614 pp. Fr. 8.00

Fischer (P. P.). Determinanten. (Sammlung Göschen. Nr. 402.) 2te, verbesserte Auflage. Berlin, Göschen, 1912. 8vo. 136 pp. Cloth. M. 0.80

Gob (A.). See Falisse (V.).

Grüttner (A.). Die Grundlagen der Geometrographie. Leipzig, 1912. M. 0.80

Hilbert (D.). Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen. (Fortschritte der mathematischen Wissenschaften in Monographien. 3tes Heft.) Leipzig, Teubner, 1912. 8vo. 26+282 pp. Cloth. M. 12.00

Hochmut (L.). Ueber ein Konoid vierten Grades. (Progr.) Wien, 1912. 8vo. 16 pp.

Kiebel (A.). Einleitung in die goniometrischen Gleichungen. (Progr.) Mies, 1911. 8vo. 5 pp.

LAPLACE (P. S.). Oeuvres complètes de Laplace, publiées sous les auspices de l'Académie des sciences, par MM. les secrétaires perpétuels. Tome 14: Correspondance et mémoires diverses. Tables générales. Nouvelle édition avec un beau portrait de Laplace. Paris, Gauthier-Villars, 1912. 4to. 8+466 pp. Fr. 20.00

LEGRAND (E.). Sommations par une formule d'Euler. (En langues française et espagnole.) Paris, 1911. 8vo. 46 pp. Fr. 3.00

LIBER MEMORIALIS. Manifestation en l'honneur de M. J. Neuberg, professeur émérite de la faculté des sciences de l'université de Liège. Liège, 1911.

LONY (G.). Einführung in die Integralrechnung im Schulunterricht. (Progr.) Hamburg, 1912. 8vo. 24 pp.

NAGER (J.). Einführung in die Elemente der geometrischen Analyse (Progr.) Klosterneuberg, 1912. 8vo. 35 pp.

NEUBERG (J.). See LIBER MEMORIALIS.

NOIRCARME (A. de). Quatrième dimension. (Editions théosophiques.) Paris, 1912. 16mo. 122 pp. Fr. 2.50

NOTICE sur les travaux scientifiques de M. Emile Borel. Paris, Gauthier-Villars, 1912. 4to. 79 pp.

NUBER (A.). Untersuchung der Kernkurven spezieller ebener Korrelationen und der damit verbundenen quadratischen Verwandtschaften. (Diss.) München, 1912. 8vo. 44 pp.

PADOA (A.). La logique déductive dans la dernière phase de son développement. Avec une préface de G. Peano. Paris, Gauthier-Villars, 1912. 8vo. 106 pp. Fr. 3.25

PALM (F. W.). Ueber die Uniformisierung von allgemeinen analytischen Funktionen. (Progr.) Wien, 1912. 8vo. 28 pp.

PASCH (M.). Vorlesungen über Geometrie. 2te, mit Zusätzen versehene Ausgabe. Leipzig, Teubner, 1912. 8vo. 4+227 pp. Cloth. M. 7.00

POINCARÉ (H.), PERRIER (E.) et PAINLEVÉ (P.). Ce que disent les choses. Paris, Hachette, 1912. 8vo. 111 pp.

ROSE (M.). Einleitung in die Funktionentheorie. Theorie der komplexen Zahlenreihen. (Sammlung Göschen.) Leipzig, Göschen, 1912. M. 0.80

SCHRUTKA EDLER v. RECHTENSTAMM (L.). Elemente der höheren Mathematik. Wien, Deuticke, 1912. 8vo. 24+569 pp. M. 10.00

SCHUR (F.). Lehrbuch der analytischen Geometrie. 2te, verbesserte und vermehrte Auflage. Leipzig, Veit, 1912. 8vo. 12+248 pp. Cloth. M. 7.50

SILBERBAUER (A.). Zum parabolischen Schnitt des Kegels. (Progr.) Waidhofen, 1912. 8vo. 2 pp.

SPORER (B.). Niedere Analysis. 2te Auflage. 4ter Abdruck. Berlin, 1912. 8vo. 179 pp. Cloth. M. 0.80

VALLÉE POUSSIN (C. J. de la). Cours d'analyse infinitésimale, Tome II. 2e édition, considérablement remaniée. Paris, Gauthier-Villars, 1912. 9+464 pp. Fr. 15.00

WINTER (M.). La méthode dans la philosophie des mathématiques. (Bibliotèque de philosophie contemporaine.) Paris, Alcan, 1911. 16mo. 200 pp. Fr. 2.50

WITTING (A.). Einführung in die Infinitesimalrechnung. (Mathematische Bibliothek, Nr. 9.) Leipzig, Teubner, 1912. 8vo. 4+73 pp. M. 0.80

ZUPANEC (J.). Ueber die Transformation der Gleichung einer Geraden in die Normalform. (Progr.) Brünn, 1912. 8vo. 5 pp.

II. ELEMENTARY MATHEMATICS.

ALGÈBRE. Cours élémentaire. Par F. G. M. 7e édition. Paris, Gigord, 1913. 8vo. 8+568 pp.

——. Cours élémentaire. 4e édition. Paris, Vitte, 1912. 16mo. 216 pp.

——. Premières notions. Par une réunion de professeurs. Paris, Poussielgue. 16mo. 88 pp.

BARNARD (S.) and CHILD (J. M.). A new algebra. Two parts. London, Macmillan, 1912. 12mo. 3s.

—— — ——. A new geometry. Part 1: Equivalent to Euclid, Book 1. London, Macmillan, 1912. 12mo. 236 pp. 1s. 6d.

BOURLET (C.). Cours abrégé de géométrie plane. Avec la collaboration de P. Baudoin. Paris, Hachette, 1912. 16mo. 309 pp. Fr. 2.50

——. Précis d'algèbre. Avec la collaboration de J. Hulot. 4e édition, revue. Paris, Hachette, 1912. 16mo. 511 pp. Fr. 3.50

BOUVART (C.) et RATINET (A.). Nouvelles tables de logarithmes à cinq décimales. 10e édition. Paris, Hachette, 1912. 8vo. 178+46 pp. Fr. 3.00

CAMMAN (P.) et RÉBOUIS (A. G.). Géométrie dans l'espace. Classes de 1re C et D. Paris, Gigord, 1912. 8vo. 204 pp.

CARONNET (T.). Cours de trigonométrie. Paris, Gauthier-Villars, 1912. 8vo. 4+217 pp. Fr. 4.50

CÉSAR (J.). Exercices d'algèbre à l'usage des écoles moyennes. Bruxelles, Dewit, 1912. 8+329 pp. Fr. 7.50

CHARLES (F.) and SUTTON (W.). Examples in elementary trigonometry. London, Christophers, 1912. 16mo. Limp. 1s.

CHILD (J. M.). See BARNARD (S.).

CIVIL service mathematics. Algebra and geometry. New regulations. London, Penningtons, 1912. 8vo. 92 pp. Boards. 1s. 6d.

COMBEROUSSE (C. de). See ROUCHÉ (E.).

COMMISSAIRE (H.). Leçons de trigonométrie. Classes de 1re C et D. Paris, Masson, 1913. 8vo. 8+273 pp. Fr. 3.00

CRANTZ (P.). Arithmetik und Algebra. 1ter Teil. (Aus Natur und Geisteswelt. Nr. 120.) 3te Auflage. Leipzig, Teubner, 1912. 8vo. 4+120 pp. Cloth. M. 1.25

DALLE (A.). 2000 théorèmes et problèmes de géométrie, avec solutions. Namur, 1912. 8vo. 8+825 pp. Fr. 16.00

DEVILLER und SCHWEIKERT. Geometrie für Mittelschulen. Strassburg, Bull, 1912. 8vo. 6+125 pp. M. 1.80

—— und SPAHN. Algebra für Mittelschulen. 2te Auflage. Strassburg, Bull, 1912. 8vo. 6+100 pp. M. 1.20

FÉAUX (B.). Buchstabenrechnung und Algebra. 12te Auflage, besorgt von F. Busch. Paderborn, Schoningh, 1912. 8vo. 7+360 pp. M. 3.80

FERVAL (H.). Eléments de trigonométrie. 6e édition, revisée. Paris, Hachette, 1912. 16mo. 297 pp. Fr. 2.50

GEBHARDT (M.). Die Geschichte der Mathematik im Mathematischen Unterrichte der höheren Schulen Deutschlands. (Abhandlungen, 3ter Band, 6tes Heft.) Leipzig, Teubner, 1912. 8vo. 7+157 pp. M. 4.80

GEIPEL (G.) und WEGEMANN (G.). Lehrbuch der Mathematik und Aufgabensammlung. 1ter Teil: Klasse III. Biclefeld, Velhagen und Klasing, 1912. 8vo. 5+126 pp. M. 1.70

GÉOMÉTRIE. Cours élémentaire. Par une réunion de professeurs. Paris, Gigord, 1912. 16mo. 112 pp.

——. Cours supérieur. Par une réunion de professeurs. Paris, Gigord, 1912. 16mo. 8+323 pp.

GROODT (G. De). Algèbre; exercices. Mons, Delporte, 1912. 75 pp. Fr. 2.00

HAMMER (E.). Mess- und Rechenübungen zur praktischen Geometrie. Ausgabe A. 5te Auflage. Stuttgart, 1912. M. 4.00

HOMANN (C.). Planimetrische Konstruktionsaufgaben. Berlin-Schöneberg, Mentor-Verlag, 1912. 52 pp. M. 1.00

HOSSFELD (C.). Der mathematische Unterricht an den höheren Schulen in den thüringischen Staaten. (Abhandlungen, 2ter Band, 6tes Heft.) Leipzig, Teubner, 1912. 8vo. 4+18 pp. M. 0.80

KETT (A.). Auflösungen für die Trigonometrie. 3te Auflage. Leipzig, Bange, 1912. 8vo. 19 pp. M. 0.60

——. Die Flächen- und Körperberechnungen. 3te Auflage. Leipzig, Bange, 1912. 8vo. 19 pp. M. 0.60

LIETZMANN (W.). Stoff und Methode des Raumlehreunterrichts in Deutschland. Ein Literaturbericht. (Abhandlungen, 5ter Band, 2tes Heft.) Leipzig, Teubner, 1912. 8vo. 8+88 pp. M. 2.80

MÜLLER (J. H. T.). Vierstellige Logarithmen. 2te, verbesserte Auflage. Halle, Waisenhaus, 1912. 8vo. 12+30 pp. M. 1.00

PENNDORF (B.). Rechnen und Mathematik im Unterricht der kaufmännischen Lehranstalten. (Abhandlungen, 4ter Band, 6tes Heft.) Leipzig, Teubner, 1912. 8vo. 6+100 pp. M. 3.00

PETERS (J.). Fünfstellige Logarithmentafel. Berlin, Reimer, 1912. 8vo. 4+83 pp. M. 7.00

PETERS (P.). Grundlagen der Arithmetik, insbesondere der Begriff der natürlichen Zahlen für den Unterricht in der Prima. (Progr.) Königsberg, 1912. 8vo. 13 pp.

RAABE (E.). Lehrbuch der Algebra für den Unterricht in den technischen Hochschulen. Bremerhaven, 1912. 8vo. Boards.

RAPPORTS sur l'enseignement des mathématiques dans les écoles, les athénées et les collèges belges. Bruxelles, Goemaere, 1911. 348 pp. Fr. 5.00

RATINET (A.). See BOUVART (C.).

RÉBOUIS (A. G.). See CAMMAN (P.).

REDGROVE (H. S.). Experimental mensuration. An elementary textbook of inductive geometry. London, Heinemann, 1912. 12mo. 346 pp. 2s. 6d.

ROUCHÉ (E.) et COMBEROUSSE (C. de). Traité de géométrie. 8e édition. 1re partie: Géométrie plane. Paris, Gauthier-Villars, 1912. 8vo. 42+547 pp. Fr. 7.50

SCHMIDT (J.). Lehrbuch der Elementarmathematik. 2ter Band. Wien, Hölder, 1912. 8vo. 5+335 pp. Cloth. M. 2.80

SICRE (C.). Des racines carrées et cubiques et des logarithmes. Paris, 1912. 8vo. Cloth. Fr. 2.50

SIMON (G.). Petit dictionnaire de calculs. Verviers, Walthère-Lejeune, 1912. 99+17+26 pp. Cloth. Fr. 5.00

SOEUR (J.). Notions d'algèbre. 2e édition. Charleroi, Fournier, 1912. 327 pp. Boards. Fr. 2.60

STUYVAERT (M.). Les nombres positifs. Manuel d'arithmétique élémentaire. 2e édition, revue. Gand, Van Goethem, 1912. 14+171 pp. Fr. 3.00

SUTTON (W.). See CHARLES (F.).

TAUBE (G.). Schemata von Determinationen geometrischer Konstruktionsaufgaben. (Progr.) Naumburg, 1911. 8vo. 60 pp.

UNTERRICHT, Der mathematische, an den Volksschulen und Lehrer- und Lehrerinnenbildungsanstalten in Süddeutschland. (Abhandlungen, 5ter Band, 3tes Heft.) Leipzig, Teubner, 1912. 8vo. 14+163 pp. M. 5.00

WEGEMANN (G.). See GEIPEL (G.).

WERNICKE (A.). Mathematik und philosophische Propädeutik. (Abhandlungen, 3ter Band, 7tes Heft.) Leipzig, Teubner, 1912. 8vo. 7+138 pp. M. 4.00

WITTMAN (W.). Funktionen und graphische Darstellungen für den neueren Arithmetikunterricht. Berlin, Göschen, 1912. M. 1.20

III. APPLIED MATHEMATICS.

ALEXANDER (W.). Columns and struts; theory and practical design. London, 1912. 8vo. 280 pp. Cloth. $4.00

ANDREWS (E. S.). Elementary principles of reinforced concrete construction. London, 1912. 8vo. 210 pp. Cloth. $1.25

BOLTZMANN (L.). Vorlesungen über Gastheorie. 2ter Teil. 2ter Abdruck. Leipzig, Barth, 1912. 10+265 pp. M. 8.00

BONNEAU (J. A.). Instruments de pesage à libre suspension. Théorie générale. Paris, Gauthier-Villars, 1911. 8vo. 94 pp. Fr. 3.50

BOUGUERET (A. et P.). Cours de dessin géométrique. 7e et 8e parties: Théorie des ombres. Paris, Monrocq, 1912. 10+8 pp.

BOULIGAND (C.). Sur les équations des petits mouvements de surface d'un fluide parfait. Paris, 1912. 8vo. 37 pp.

BOUSSINESQ (J.). Théorie géométrique, pour un corps non rigide, des déplacements bien continus, ainsi que des déformations et des rotations de ses particules. Paris, Gauthier-Villars, 1912. 4to. 17 pp. Fr. 1.00

BRIGGS (H.). The effects of errors in surveying. London, 1912. 8vo. 190 pp. Cloth. $1.75

CAPART (G.). See VIEWEGER (H.).

CRÉPIN de BEAUREGARD (P.). Guide scientifique du géographe-explorateur. (Ouvrage courronné.) Paris, Gauthier-Villars, 1912. 8vo. 10+250 pp. Fr. 10.00

DRUDE. Physik des Aethers. 2te Auflage, von W. König. Stuttgart, Enke, 1912. M. 16.00

EBNER (F.). Technische Infinitesimalrechnung. Berlin, Salle, 1912. 8vo. M. 2.40

GALLE (A.). Mathematische Instrumente. Rechenschieber, Rechenmaschinen, Kurven- und Flächenmesser, Integratoren, Analysatoren und Integraphen. (Mathematisch-physikalische Schriften, Nr. 15.) Leipzig, Teubner, 1912. 8vo. 4+187 pp. Cloth. M. 4.80

GÉRARD (E.). Mesures électriques. 4e édition. Paris, Gauthier-Villars, 1912. 8+749 pp. Fr. 12.00

GUILLET (A.). Propriétés cinématiques fondamentales des vibrations. Notes de M. Aubert. Paris, Gauthier-Villars, 1913. 8vo. 410 pp. Fr. 16.00

HOSKINS (L. M.). Theoretical mechanics; an elementary text-book. 4th edition. Stanford University, Cal., 1911. 8vo. 467 pp. Cloth. $3.50

HOUSTON (R. A.). An introduction to mathematical physics. London, Longmans, 1912. 12mo. 210 pp. 6s.

ISRAEL (O.). Zur Theorie der einseitig wirkenden Instrumentalfehler an Repetitionstheodoliten. Dresden, 1912. 8vo. 69 pp. M. 2.20

JAMIESON (A.). A text-book of applied mechanics and mechanical engineering. Volume 1. 9th edition, thoroughly revised and enlarged. London, Griffin, 1912. 12mo. 430 pp. 6s.

KÖNIG (W.). See DRUDE.

LARMOR (Sir J.). See THOMSON (J.).

MACDONALD (H. M.). The effect produced on an obstacle by a train of electrical waves. London, Dulau, 1912. 4to. 39 pp. Sewed. 2s.

MACH (E.). Die Mechanik in ihrer Entwicklung. Historisch-kritisch dargestellt. 7te, verbesserte und vermehrte Auflage. Leipzig, 1912. 8vo. 12+494 pp. M. 8.00

METH (P.). Theorie der Planetenbewegung. (Mathematische Bibliothek, Nr. 8.) Leipzig, Teubner, 1912. 8vo. 4+60 pp. M. 0.80

MICHAELIS (L.). Einführung in die Mathematik für Biologen und Chemiker. Berlin, Springer, 1912. 8vo. 6+253 pp. Cloth. M. 7.80

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